AN INVOLUTION OF PERIOD SEVENTEEN

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TABLE OF CONTENTS

TABLE OF CONTENTS--(Continued)

CHAPTER I

INTRODUCTION

In an extended complex plane with homogeneous coordinates the equations

$$x_1' : x_2' : x_3' = x_1 : Ex_2 : E^{\alpha}x_3$$

define a plane cyclic homography of period p, where p is a prime number greater than two, E a pth primitive root of unity and α is an integer greater than unity and less than p. This homography generates an involution of period p.

Lucien Godeaux has been the world's leader in studying involutions. Since his paper in 1916 [5] where he used period three, he has published many papers on involutions. Many other authors have contributed to this field. Hutcherson studied involutions of period seven and eleven [14, 15], Childress studied some of period three, five, and thirteen [18, 19], Frank studied some of period eleven [3], and Gormsen studied some of period three, five, and seven [12].

This writer is investigating the mapping of an involution of period seventeen from a plane onto a surface in a space of ten dimensions (S_{10}). The three branch points of this surface ϕ require detailed study comprising Chapter II. In Chapter III certain projections of ϕ are investigated. A rational surface F, in S_{11}, is exhibited in Chapter IV. Points on this surface are in a one-to-one correspondence with points

on the original plane, whereas points of the surface Φ are in a one-to-seventeen correspondence with points on the plane as well as with points on surface F.

The material in Chapter II was the subject of a joint paper given at the 1960 summer meeting of the American Mathematical Society [22]. The contents of Chapter IV were used in another joint paper given also at this meeting [21].

The reader is referred to the bibliography for introductory material to this area. One unfamiliar with the usage of terms, symbols, and techniques of this phase of Algebraic Geometry might not fully understand certain areas of this dissertation, e.g., first order neighborhoods [12]. Also homogeneous projective coordinates are used exclusively [13]. Since introductory material is plentiful and available it is usually omitted from most areas and references are mentioned instead.

As far as the author has been able to determine, most of this work is original.

CHAPTER II

A SURFACE ϕ OBTAINED FROM AN INVOLUTION OF PERIOD SEVENTEEN

1. The Image Surface ϕ

Consider the homography,

$$(H) \qquad x_1' : x_2' : x_3' = x_1 : Ex_2 : E^{15}x_3$$

where E is a primitive seventeenth root of unity. This homography generates an involution, I_{17}, of period seventeen. A group of I_{17} is composed of the following seventeen points,

$$(x_1, x_2, x_3), (x_1, Ex_2, E^{15}x_3), (x_1, E^2x_2, E^{13}x_3),$$
$$(x_1, E^3x_2, E^{11}x_3), (x_1, E^4x_2, E^9x_3), (x_1, E^5x_2, E^7x_3),$$
$$(x_1, E^6x_2, E^5x_3), (x_1, E^7x_2, E^3x_3), (x_1, E^8x_2, Ex_3),$$
$$(x_1, E^9x_2, E^{16}x_3), (x_1, E^{10}x_2, E^{14}x_3), (x_1, E^{11}x_2, E^{12}x_3),$$
$$(x_1, E^{12}x_2, E^{10}x_3), (x_1, E^{13}x_2, E^8x_3), (x_1, E^{14}x_2, E^6x_3),$$
$$(x_1, E^{15}x_2, E^4x_3), (x_1, E^{16}x_2, E^2x_3).$$

Now consider the complete non-invariant linear system of order seventeen in the plane, i.e.,

$$\Sigma\ a_h\ x_1^i\ x_2^j\ x_3^k = 0$$

where $i + j + k = 17$ and $h = 1, 2, \ldots, 171$ designates the different coefficients. In this general system there are seventeen systems of curves that are transformed into themselves by the homography H.

$$(1)\quad a_1 x_1^{17} + a_9 x_1^{14} x_2^2 x_3 + a_{26} x_1^{11} x_2^4 x_3^2 + a_{52} x_1^8 x_2^6 x_3^3 + a_{59} x_1^7 x_2 x_3^9 + a_{87} x_1^5 x_2^8 x_3^4 + a_{99} x_1^4 x_2^3 x_3^{10} + a_{133} x_1^2 x_2^{10} x_3^5 + a_{148} x_1 x_2^5 x_3^{11} + a_{170} x_2^{17} + a_{171} x_3^{17} = 0.$$

$$(2)\quad g(x_1, x_2, x_3) = a_{17} x_1^{12} x_3^5 + a_{29} x_1^{10} x_2^7 + a_{42} x_1^9 x_2^2 x_3^6 + a_{58} x_1^7 x_2^9 x_3 + a_{76} x_1^6 x_2^4 x_3^7 + a_{96} x_1^4 x_2^{11} x_3^2 + a_{119} x_1^3 x_2^6 x_3^8 + a_{124} x_1^2 x_2 x_3^{14} + a_{143} x_1 x_2^{13} x_3^3 + a_{169} x_2^8 x_3^9 = 0.$$

$$(3)\quad a_{10} x_1^{14} x_2 x_3^2 + a_{28} x_1^{11} x_2^3 x_3^3 + a_{54} x_1^8 x_2^5 x_3^4 + a_{57} x_1^7 x_3^{10} + a_{89} x_1^5 x_2^7 x_3^5 + a_{97} x_1^4 x_2^2 x_3^{11} + a_{106} x_1^3 x_2^{14} + a_{133} x_1^2 x_2^9 x_3^6 + a_{146} x_1 x_2^4 x_3^{12} + a_{154} x_2^{16} x_3 = 0.$$

$$(4)\quad a_{11} x_1^{13} x_2^4 + a_{31} x_1^{10} x_2^6 x_3 + a_{40} x_1^9 x_2 x_3^7 + a_{60} x_1^7 x_2^8 x_3^2 + a_{74} x_1^6 x_2^3 x_3^8 + a_{98} x_1^4 x_2^{10} x_3^3 + a_{117} x_1^3 x_2^5 x_3^9 + a_{122} x_1^2 x_3^{15} + a_{145} x_1 x_2^{12} x_3^4 + a_{167} x_2^7 x_3^{10} = 0.$$

(5) $$a_8 x_1^{14} x_3^3 + a_{27} x_1^{11} x_2^2 x_3^4 + a_{55} x_1^8 x_2^4 x_3^5 + a_{67} x_1^6 x_2^{11} + a_{91} x_1^5 x_2^6 x_3^6 + a_{95} x_1^4 x_2 x_3^{12} + a_{108} x_1^3 x_2^{13} x_3 + a_{135} x_1^2 x_2^8 x_3^7 + a_{144} x_1 x_2^3 x_3^{13} + a_{156} x_2^{15} x_3^2 = 0.$$

(6) $$a_2 x_1^{16} x_2 + a_{13} x_1^{13} x_2^3 x_3 + a_{33} x_1^{10} x_2^5 x_3^2 + a_{38} x_1^9 x_3^8 + a_{62} x_1^7 x_2^7 x_3^3 + a_{72} x_1^6 x_2^2 x_3^9 + a_{100} x_1^4 x_2^9 x_3^4 + a_{115} x_1^3 x_2^4 x_3^{10} + a_{147} x_1 x_2^{11} x_3^5 + a_{165} x_2^6 x_3^{11} = 0.$$

(7) $$a_{25} x_1^{11} x_2 x_3^5 + a_{37} x_1^9 x_2^8 + a_{53} x_1^8 x_2^3 x_3^6 + a_{69} x_1^6 x_2^{10} x_3 + a_{90} x_1^5 x_2^5 x_3^7 + a_{93} x_1^4 x_3^{13} + a_{110} x_1^3 x_2^{12} x_3^2 + a_{136} x_1^2 x_2^7 x_3^8 + a_{142} x_1^1 x_2^2 x_3^{14} + a_{158} x_2^{14} x_3^3 = 0.$$

(8) $$a_3 x_1^{16} x_3 + a_{15} x_1^{13} x_2^2 x_3^2 + a_{35} x_1^{10} x_2^4 x_3^3 + a_{64} x_1^7 x_2^6 x_3^4 + a_{70} x_1^6 x_2 x_3^{10} + a_{102} x_1^4 x_2^8 x_3^5 + a_{113} x_1^3 x_2^3 x_3^{11} + a_{121} x_1^2 x_2^{15} + a_{149} x_1 x_2^{10} x_3^6 + a_{163} x_2^5 x_3^{12} = 0.$$

(9) $$a_{16} x_1^{12} x_2^5 + a_{23} x_1^{11} x_3^6 + a_{39} x_1^9 x_2^7 x_3 + a_{51} x_1^8 x_2^2 x_3^7 + a_{71} x_1^6 x_2^9 x_3^2 + a_{88} x_1^5 x_2^4 x_3^8 + a_{112} x_1^3 x_2^{11} x_3^3 + a_{134} x_1^2 x_2^6 x_3^9 + a_{140} x_1 x_2 x_3^{15} + a_{160} x_2^{13} x_3^4 = 0.$$

(10) $$a_{14}x_1^{13}x_2x_3^3 + a_{36}x_1^{10}x_2^3x_3^4 + a_{66}x_1^7x_2^5x_3^5 + a_{68}x_1^6x_3^{11}$$
$$+ a_{79}x_1^5x_2^{12} + a_{104}x_1^4x_2^7x_3^6 + a_{111}x_1^3x_2^2x_3^{12} + a_{123}x_1^2x_2^{14}x_3$$
$$+ a_{151}x_1x_2^9x_3^7 + a_{161}x_2^4x_3^{13} = 0.$$

(11) $$a_4x_1^{15}x_2^2 + a_{18}x_1^{12}x_2^4x_3 + a_{41}x_1^9x_2^6x_3^2 + a_{49}x_1^8x_2x_3^8$$
$$+ a_{73}x_1^6x_2^8x_3^3 + a_{86}x_1^5x_2^3x_3^9 + a_{114}x_1^3x_2^{10}x_3^4 + a_{132}x_1^2x_2^5x_3^{10}$$
$$+ a_{138}x_1x_3^{16} + a_{162}x_2^{12}x_3^5 = 0.$$

(12) $$a_{12}x_1^{13}x_3^4 + a_{34}x_1^{10}x_2^2x_3^5 + a_{46}x_1^8x_2^9 + a_{65}x_1^7x_2^4x_3^6$$
$$+ a_{81}x_1^5x_2^{11}x_3 + a_{105}x_1^4x_2^6x_3^7 + a_{109}x_1^3x_2x_3^{13} + a_{125}x_1^2x_2^{13}x_3^2$$
$$+ a_{153}x_1x_2^8x_3^8 + a_{159}x_2^3x_3^{14} = 0.$$

(13) $$a_6x_1^{15}x_2x_3 + a_{20}x_1^{12}x_2^3x_3^2 + a_{43}x_1^9x_2^5x_3^3 + a_{47}x_1^8x_3^9$$
$$+ a_{75}x_1^6x_2^7x_3^4 + a_{84}x_1^5x_2^2x_3^{10} + a_{116}x_1^3x_2^9x_3^5 + a_{130}x_1^2x_2^4x_3^{11}$$
$$+ a_{137}x_1x_2^{16} + a_{164}x_2^{11}x_3^6 = 0.$$

(14) $$a_{22}x_1^{11}x_2^6 + a_{32}x_1^{10}x_2x_3^6 + a_{48}x_1^8x_2^8x_3 + a_{63}x_1^7x_2^3x_3^7$$
$$+ a_{83}x_1^5x_2^{10}x_3^2 + a_{103}x_1^4x_2^5x_3^8 + a_{107}x_1^3x_3^{14} + a_{127}x_1^2x_2^{12}x_3^3$$
$$+ a_{152}x_1x_2^7x_3^9 + a_{157}x_2^2x_3^{15} = 0.$$

$$(15) \quad a_5 x_1^{15} x_3^2 + a_{21} x_1^{12} x_2^2 x_3^3 + a_{45} x_1^9 x_2^4 x_3^4 + a_{77} x_1^6 x_2^6 x_3^5 + a_{82} x_1^5 x_2 x_3^{11} + a_{92} x_1^4 x_2^{13} + a_{118} x_1^3 x_2^8 x_3^6 + a_{128} x_1^2 x_2^3 x_3^{12} + a_{139} x_1 x_2^{15} x_3 + a_{166} x_2^{10} x_3^7 = 0.$$

$$(16) \quad a_7 x_1^{14} x_2^3 + a_{24} x_1^{11} x_2^5 x_3 + a_{30} x_1^{10} x_3^7 + a_{50} x_1^8 x_2^7 x_3^2 + a_{61} x_1^7 x_2^2 x_3^8 + a_{85} x_1^5 x_2^9 x_3^3 + a_{101} x_1^4 x_2^4 x_3^9 + a_{129} x_1^2 x_2^{11} x_3^4 + a_{150} x_1 x_2^6 x_3^{10} + a_{155} x_2 x_3^{16} = 0.$$

$$(17) \quad a_{19} x_1^{12} x_2 x_3^4 + a_{44} x_1^9 x_2^3 x_3^5 + a_{56} x_1^7 x_2^{10} + a_{78} x_1^6 x_2^5 x_3^6 + a_{80} x_1^5 x_3^{12} + a_{94} x_1^4 x_2^{12} x_3 + a_{120} x_1^3 x_2^7 x_3^7 + a_{126} x_1^2 x_2^2 x_3^{13} + a_{141} x_1 x_2^{14} x_3^2 + a_{168} x_2^9 x_3^8 = 0.$$

Now relate the curves of system (1) projectively to the hyperplanes of S_{10} by taking the projective transformation

$$(T) \quad \frac{X_1}{x_1^{17}} = \frac{X_2}{x_1^{14} x_2^2 x_3} = \frac{X_3}{x_1^{11} x_2^4 x_3^2} = \frac{X_4}{x_1^8 x_2^6 x_3^3} = \frac{X_5}{x_1^7 x_2 x_3^9} = \frac{X_6}{x_1^5 x_2^8 x_3^4} = \frac{X_7}{x_1^4 x_2^3 x_3^{10}} = \frac{X_8}{x_1^2 x_2^{10} x_3^5} = \frac{X_9}{x_1 x_2^5 x_3^{11}} = \frac{X_{10}}{x_2^{17}} = \frac{X_{11}}{x_3^{17}} .$$

Eliminate x_1, x_2, and x_3 in T to obtain a surface ϕ in S_{10} which has for its equations

$$(\phi) \qquad \left\| \begin{matrix} X_{10}X_1 & X_8X_5 & X_6 & X_4 & X_9X_5 & X_3 & X_7 & X_2X_5 & X_2^2 \\ X_4X_6 & X_7X_9 & X_9 & X_7 & X_7X_{11} & X_5 & X_{11} & X_1X_{11} & X_1X_5 \end{matrix} \right\| = 0.$$

This surface is the image of I_{17}, i.e., a set of I_{17} corresponds to a single point of the surface ϕ. Now investigate the singularities of the surface ϕ at the images of the fixed points of I_{17}. The images of O_1 (1, 0, 0), O_2 (0, 1, 0), and O_3 (0, 0, 1) are O_1' (1,0,0,0,0,0,0,0,0,0,0,0), O_{10}' (0,0,0,0,0,0,0,0,0,1,0), and O_{11}' (0,0,0,0,0,0,0,0,0,0,1) respectively.

2. Branch Point O_1'

This investigation is based on a technique that finds the projective images of successive neighborhoods of the vertices of the triangle of reference in the plane. This definition of neighborhood is based on the existence of a quadratic transformation which relates two planes with homogeneous coordinates in such a manner that a reference triangle vertex in the z coordinate plane is mapped onto its corresponding reference triangle vertex in the x coordinate plane, but the x coordinate vertex is mapped to the meaningless point (0, 0, 0) in the z plane. For example, O_1 (1, 0, 0) in the z coordinate plane is mapped onto O_1 (1, 0, 0) in the x coordinate plane,

and O_1 (1, 0, 0) in the x plane is mapped onto (0, 0, 0) in the z coordinate plane. Then the z plane image of the point $S(1 + \lambda\alpha, \lambda\beta, \lambda\gamma)$ from the x plane, will be the image of the first order neighborhood of O_1 if the limit is taken as λ tends to zero. For a more detailed discussion of neighborhoods the reader is referred to Morelock [23].

Note that members of the family (1) do not in general go through the vertices of the reference triangle. But if the restriction $a_1 = 0$ is added, a new family (18) will go through.

$$(18) \qquad a_9 x_1^{14} x_2^2 x_3 + a_{26} x_1^{11} x_2^4 x_3^2 + a_{52} x_1^8 x_2^6 x_3^3 + a_{59} x_1^7 x_2 x_3^9 + a_{87} x_1^5 x_2^8 x_3^4 + a_{99} x_1^4 x_2^3 x_3^{10} + a_{133} x_1^2 x_2^{10} x_3^5 + a_{148} x_1 x_2^5 x_3^{11} + a_{170} x_2^{17} + a_{171} x_3^{17} = 0.$$

The quadratic transformation R and its inverse will relate P_1 (1, 0, 0) in the z coordinate plane to O_1 (1, 0, 0) in the x coordinate plane and O_1 (1, 0, 0) to the meaningless point (0, 0, 0).

$$(R) \qquad x_1 : x_2 : x_3 = z_1^2 : z_1 z_2 : z_2 z_3$$

$$(R)^{-1} \qquad z_1 : z_2 : z_3 = x_1 x_2 : x_2^2 : x_1 x_3$$

Apply the transformation R successively fourteen times to equation (18) and arrive at

$$(19) \qquad z_1^{238} (a_{170} z_2 + a_9 z_3) + a_{171} z_2^{222} z_3^{17} + a_{59} z_1^{119} z_2^{111} z_3^9 + a_{26} z_1^{221} z_2^{16} z_3^2 + a_{99} z_1^{102} z_2^{127} z_3^{10} + a_{52} z_1^{204} z_2^{32} z_3^3 + a_{87} z_1^{187} z_2^{48} z_3^4 + a_{148} z_1^{85} z_2^{143} z_3^{11} + a_{133} z_1^{170} z_2^{64} z_3^5 = 0.$$

This shows that the point $(z_2 = z_3 = 0)$, corresponds to the point in the fourteenth order neighborhood of O_1 in the direction of $x_3 = 0$, i.e., $O_{122222222222222} = O_{12(14)}$ [3, p. 32].

Now apply the transformation, R, fourteen successive times to the transformation

$$(20)\qquad \frac{X_2}{x_1^{14}\,x_2^{2}\,x_3} = \frac{X_3}{x_1^{11}\,x_2^{4}\,x_3^{2}} = \frac{X_4}{x_1^{8}\,x_2^{6}\,x_3^{3}} = \frac{X_5}{x_1^{7}\,x_2\,x_3^{9}}$$

$$= \frac{X_6}{x_1^{5}\,x_2^{8}\,x_3^{4}} = \frac{X_7}{x_1^{4}\,x_2^{3}\,x_3^{10}} = \frac{X_8}{x_1^{2}\,x_2^{10}\,x_3^{5}} = \frac{X_9}{x_1\,x_2^{5}\,x_3^{11}}$$

$$= \frac{X_{10}}{x_2^{17}} = \frac{X_{11}}{x_3^{17}}\,.$$

The simplified result is

$$(21)\qquad \frac{X_2}{z_1^{238}\,z_3} = \frac{X_3}{z_1^{221}\,z_2^{16}\,z_3^{2}} = \frac{X_4}{z_1^{204}\,z_2^{32}\,z_3^{3}} = \frac{X_5}{z_1^{119}\,z_2^{111}\,z_3^{9}}$$

$$= \frac{X_6}{z_1^{187}\,z_2^{48}\,z_3^{4}} = \frac{X_7}{z_1^{102}\,z_2^{127}\,z_3^{10}} = \frac{X_8}{z_1^{170}\,z_2^{64}\,z_3^{5}}$$

$$= \frac{X_9}{z_1^{85}\,z_2^{143}\,z_3^{11}} = \frac{X_{10}}{z_1^{238}\,z_2} = \frac{X_{11}}{z_2^{222}\,z_3^{17}}\,.$$

A substitution of $z_3 = k\, z_2$ will allow an all directional approach to the point $(z_2 = z_3 = 0)$. This substitution in (21), after simplification, gives

$$(22)\qquad \frac{X_2}{k\, z_1^{238}} = \frac{X_3}{k^2\, z_1^{221}\, z_2^{17}} = \frac{X_4}{k^3\, z_1^{204}\, z_2^{34}} = \frac{X_5}{k^9\, z_1^{119}\, z_2^{119}}$$

$$= \frac{X_6}{k^4\, z_1^{187}\, z_2^{51}} = \frac{X_7}{k^{10}\, z_1^{102}\, z_2^{136}} = \frac{X_8}{k^5\, z_1^{170}\, z_2^{68}}$$

$$= \frac{X_9}{k^{11}\, z_1^{85}\, z_2^{153}} = \frac{X_{10}}{z_1^{238}} = \frac{X_{11}}{k^{17}\, z_2^{238}}\ .$$

As z_2 tends to the limit zero, the above gives the equation of a plane tangent to ϕ at O_1'. It is

$$(23)\qquad \begin{cases} X_2 = k\, X_{10} \\ X_3 = X_4 = X_5 = X_6 = X_7 = X_8 = X_9 = X_{11} = 0. \end{cases}$$

Now examine a different quadratic transformation and its inverse

$$(S)\qquad x_1 : x_2 : x_3 = z_1^2 : z_2 z_3 : z_1 z_3$$

$$(S)^{-1}\qquad z_1 : z_2 : z_3 = x_1 x_3 : x_1 x_2 : x_3^2\ .$$

Apply it seven times to the curve (18), and the simplified result is

$$(24) \qquad z_1^{119}\,(a_{171}\,z_3^2 + a_{59}\,z_2 z_3 + a_9\,z_2^2) + a_{170}\,z_2^{17}\,z_3^{104}$$
$$+ a_{26}\,z_1^{102}\,z_2^4\,z_3^{15} + a_{99}\,z_1^{102}\,z_2^3\,z_3^{16} + a_{52}\,z_1^{85}\,z_2^6\,z_3^{30}$$
$$+ a_{87}\,z_1^{68}\,z_2^8\,z_3^{45} + a_{148}\,z_1^{85}\,z_2^5\,z_3^{31} + a_{133}\,z_1^{51}\,z_2^{10}\,z_3^{60} = 0.$$

This shows that in the seventh order neighborhood of O_1, the point on the curve (18) along the direction $x_2 = 0$ corresponds to the double point $(z_2 = z_3 = 0)$.

Repeated application of the transformation S to the transformation (20), yields

$$(25) \qquad \frac{X_2}{z_1^{119}\,z_2^2} = \frac{X_3}{z_1^{102}\,z_2^4\,z_3^{15}} = \frac{X_4}{z_1^{85}\,z_2^6\,z_3^{30}} = \frac{X_5}{z_1^{119}\,z_2 z_3}$$
$$= \frac{X_6}{z_1^{68}\,z_2^8\,z_3^{45}} = \frac{X_7}{z_1^{102}\,z_2^3\,z_3^{16}} = \frac{X_8}{z_1^{51}\,z_2^{10}\,z_3^{60}} = \frac{X_9}{z_1^{85}\,z_2^5\,z_3^{31}}$$
$$= \frac{X_{10}}{z_2^{17}\,z_3^{104}} = \frac{X_{11}}{z_1^{119}\,z_3^2}\,.$$

Substitute $z_3 = k\,z_2$ to allow for an all directional approach to the point $(z_2 = z_3 = 0)$. This simplifies to give

$$(26)\quad \frac{X_2}{z_1^{119}} = \frac{X_3}{k^{15}\, z_1^{102}\, z_2^{17}} = \frac{X_4}{k^{30}\, z_1^{85}\, z_2^{34}} = \frac{X_5}{k\, z_1^{119}}$$

$$= \frac{X_6}{k^{45}\, z_1^{68}\, z_2^{51}} = \frac{X_7}{k^{16}\, z_1^{102}\, z_2^{17}} = \frac{X_8}{k^{60}\, z_1^{51}\, z_2^{68}}$$

$$= \frac{X_9}{k^{31}\, z_1^{85}\, z_2^{34}} = \frac{X_{10}}{k^{104}\, z_2^{119}} = \frac{X_{11}}{k^{2}\, z_1^{119}} .$$

Now let z_2 approach the limit zero, and the equation of the other tangent element to ϕ at O_1' is

$$(27)\quad \begin{cases} X_5^2 - X_{11}\, X_2 = 0 \\ X_3 = X_4 = X_6 = X_7 = X_8 = X_9 = X_{10} = 0. \end{cases}$$

This surface (27) when investigated is seen to be a quadric cone.† The reader is referred to Gormsen [12] for a different method for this investigation, i.e., Coble's method.

Hence, the following

<u>Theorem 1</u>: The tangent elements to the surface ϕ at the point O_1' (1,0,0,0,0,0,0,0,0,0,0) are a plane (23) and a quadric cone (27).

†See Appendix I.

3. Branch Point O'_{10}

The point O_2 $(0, 1, 0)$ corresponds to the point O'_{10} $(0,0,0,0,0,0,0,0,0,1,0)$ in S_{10} by the transformation T. To study the tangent elements at this point, examine the system (∞^9) of curves passing through O_2, i.e., system (1) with $a_{170} = 0$. We have

$$(28)\qquad a_1 x_1^{17} + a_9 x_1^{14} x_2^2 x_3 + a_{26} x_1^{11} x_2^4 x_3^2 + a_{52} x_1^8 x_2^6 x_3^3 + a_{59} x_1^7 x_2 x_3^9 + a_{87} x_1^5 x_2^8 x_3^4 + a_{99} x_1^4 x_2^3 x_3^{10} + a_{133} x_1^2 x_2^{10} x_3^5 + a_{148} x_1 x_2^5 x_3^{11} + a_{171} x_3^{17} = 0.$$

Apply the quadratic transformation

$$(U)\qquad x_1 : x_2 : x_3 = z_1 z_2 : z_2^2 : z_1 z_3$$

$$(U)^{-1}\qquad z_1 : z_2 : z_3 = x_1^2 : x_1 x_2 : x_2 x_3$$

twice in succession to the system (28). We get

$$(29)\qquad z_2^{34} (a_1 x_1^5 + a_9 z_1^4 z_3 + a_{26} x_1^3 x_3^2 + a_{52} z_1^2 z_3^3 + a_{87} z_1 z_3^4 + a_{133} z_3^5) + a_{171} z_1^{22} z_3^{17} + a_{59} z_1^{13} z_2^{17} z_3^9 + a_{99} z_1^{12} z_2^{17} z_3^{10} + a_{148} z_1^{11} z_2^{17} z_3^{11} = 0.$$

This indicates that O_{211}, the second order neighborhood point in the $x_3 = 0$ direction, corresponds to the five tuple point $(z_1 = z_3 = 0)$.

Now apply the transformation U twice to the transformation

$$(30)\qquad \frac{X_1}{x_1^{17}} = \frac{X_2}{x_1^{14}\,x_2^2\,x_3} = \frac{X_3}{x_1^{11}\,x_2^4\,x_3^2} = \frac{X_4}{x_1^8\,x_2^6\,x_3^3}$$

$$= \frac{X_5}{x_1^7\,x_2\,x_3^9} = \frac{X_6}{x_1^5\,x_2^8\,x_3^4} = \frac{X_7}{x_1^4\,x_2^3\,x_3^{10}} = \frac{X_8}{x_1^2\,x_2^{10}\,x_3^5}$$

$$= \frac{X_9}{x_1\,x_2^5\,x_3^{11}} = \frac{X_{11}}{x_3^{17}}\ .$$

This gives

$$(31)\qquad \frac{X_1}{z_1^5\,z_2^{34}} = \frac{X_2}{z_1^4\,z_2^{34}\,z_3} = \frac{X_3}{z_1^3\,z_2^{34}\,z_3^2} = \frac{X_4}{z_1^2\,z_2^{34}\,z_3^3}$$

$$= \frac{X_5}{z_1^{13}\,z_2^{17}\,z_3^9} = \frac{X_6}{z_1\,z_2^{34}\,z_3^4} = \frac{X_7}{z_1^{12}\,z_2^{17}\,z_3^{10}} = \frac{X_8}{z_2^{34}\,z_3^5}$$

$$= \frac{X_9}{z_1^{11}\,z_2^{17}\,z_3^{11}} = \frac{X_{11}}{z_1^{22}\,z_3^{17}}\ .$$

Let $z_1 = k\ z_3$, to allow an approach from all directions to the image point $(z_1 = z_3 = 0)$,

$$(32) \qquad \frac{X_1}{k^5\, z_2^{34}} = \frac{X_2}{k^4\, z_2^{34}} = \frac{X_3}{k^3\, z_2^{34}} = \frac{X_4}{k^2\, z_2^{34}} = \frac{X_5}{k^{13}\, z_2^{17}\, z_3^{17}}$$

$$= \frac{X_6}{k\, z_2^{34}} = \frac{X_7}{k^{12}\, z_2^{17}\, z_3^{17}} = \frac{X_8}{z_2^{34}} = \frac{X_9}{k^{11}\, z_2^{17}\, z_3^{17}} = \frac{X_{11}}{k^{22}\, z_3^{34}} .$$

As z_3 approaches the limit zero in (32), the equation of a tangent element is arrived at

$$(33) \qquad \begin{cases} \left\| \begin{matrix} X_1 & X_2 & X_3 & X_4 & X_6 \\ X_2 & X_3 & X_4 & X_6 & X_8 \end{matrix} \right\| = 0 \\ \\ X_5 = X_7 = X_9 = X_{11} = 0. \end{cases}$$

This tangent surface is verified to be a quintic cone when investigated.[†]

To study the neighborhood points along the direction $x_1 = 0$, examine (28) with the quadratic transformation

$$(V) \qquad x_1 : x_2 : x_3 = z_1 z_3 : z_2^2 : z_2 z_3$$

$$(V)^{-1} \qquad z_1 : z_2 : z_3 = x_1 x_2 : x_2 x_3 : x_3^2.$$

Apply the transformation V five successive times to equation (28); this gives

[†] See Appendix I.

$$(34) \qquad z_2^{85}\,(a_{171}\,z_3^2 + a_{148}\,z_1\,z_3 + a_{133}\,z_1^2) + a_1\,z_1^{17}\,z_3^{70}$$

$$+ a_{59}\,z_1^7\,z_2^{51}\,z_3^{29} + a_9\,z_1^{14}\,z_2^{17}\,z_3^{56} + a_{26}\,z_1^{11}\,z_2^{34}\,z_3^{42}$$

$$+ a_{99}\,z_1^4\,z_2^{68}\,z_3^{15} + a_{52}\,z_1^8\,z_2^{51}\,z_3^{28} + a_{87}\,z_1^5\,z_2^{68}\,z_3^{14} = 0.$$

This indicates that the fifth order neighborhood point $(0_{23(5)})$ in the direction $x_1 = 0$, corresponds to the double point $(z_1 = z_3 = 0)$.

Now apply the transformation V repeatedly five times to the transformation (30), and get

$$(35) \qquad \frac{X_1}{z_1^{17}\,z_3^{70}} = \frac{X_2}{z_1^{14}\,z_2^{17}\,z_3^{56}} = \frac{X_3}{z_1^{11}\,z_2^{34}\,z_3^{42}} = \frac{X_4}{z_1^{8}\,z_2^{51}\,z_3^{28}}$$

$$= \frac{X_5}{z_1^{7}\,z_2^{51}\,z_3^{29}} = \frac{X_6}{z_1^{5}\,z_2^{68}\,z_3^{14}} = \frac{X_7}{z_1^{4}\,z_2^{68}\,z_3^{14}} = \frac{X_8}{z_1^{2}\,z_2^{85}}$$

$$= \frac{X_9}{z_1\,z_2^{85}\,z_3} = \frac{X_{11}}{z_2^{85}\,z_3^{2}}\,.$$

To allow for an approach from any direction to the point $(z_1 = z_3 = 0)$, let $z_3 = k\,z_1$, and obtain

$$(36) \qquad \frac{X_1}{k^{70}\, z_1^{85}} = \frac{X_2}{k^{56}\, z_1^{68}\, z_2^{17}} = \frac{X_3}{k^{42}\, z_1^{51}\, z_2^{34}} = \frac{X_4}{k^{28}\, z_1^{34}\, z_2^{51}}$$

$$= \frac{X_5}{k^{29}\, z_1^{34}\, z_2^{51}} = \frac{X_6}{k^{14}\, z_1^{17}\, z_2^{68}} = \frac{X_7}{k^{15}\, z_1^{17}\, z_2^{68}} = \frac{X_8}{z_2^{85}}$$

$$= \frac{X_9}{k\, z_2^{85}} = \frac{X_{11}}{k^2\, z_2^{85}} .$$

As z_1 tends to the limit zero in (36) the other tangent element is given as

$$(37) \qquad \begin{cases} X_9^2 - X_8 X_{11} = 0 \\ X_1 = X_2 = X_3 = X_4 = X_5 = X_6 = X_7 = 0. \end{cases}$$

This is seen to be a quadric cone when the order is determined.

Thus,

<u>Theorem 2</u>: The tangent elements to the surface ϕ at the point O_{10}' (0,0,0,0,0,0,0,0,0,1,0) are a quintic cone (33) and a quadric cone (37).

4. Branch Point O_{11}'

The point O_3 (0, 0, 1) on the plane corresponds under the transformation T to the point O_{11}' (0,0,0,0,0,0,0,0,0,0,1). To study O_{11}' investigate the system of curves that are members of (1) and have the restriction $a_{171} = 0$. They are

(38) $$a_1 x_1^{17} + a_9 x_1^{14} x_2^2 x_3 + a_{26} x_1^{11} x_2^4 x_3^2 + a_{52} x_1^8 x_2^6 x_3^3$$
$$+ a_{59} x_1^7 x_2 x_3^9 + a_{87} x_1^5 x_2^8 x_3^4 + a_{99} x_1^4 x_2^3 x_3^{10} + a_{133} x_1^2 x_2^{10} x_3^5$$
$$+ a_{148} x_1 x_2^5 x_3^{11} + a_{170} x_2^{17} = 0.$$

Apply the transformation

(M) $$x_1 : x_2 : x_3 = z_1 z_2 : z_2 z_3 : z_3^2$$

$(M)^{-1}$ $$z_1 : z_2 : z_3 = x_1 x_3 : x_2^2 : x_2 x_3$$

eleven times in succession to (38). The result is

(39) $$z_3^{187} (a_{170} z_2 + a_{148} z_1) + a_1 z_1^{17} z_2^{171} + a_{59} z_1^7 z_2^{62} z_3^{119}$$
$$+ a_9 z_1^{14} z_2^{140} z_3^{34} + a_{26} z_1^{11} z_2^{109} z_3^{68} + a_{99} z_1^4 z_2^{31} z_3^{153}$$
$$+ a_{52} z_1^8 z_2^{78} z_3^{102} + a_{87} z_1^5 z_2^{47} z_3^{136} + a_{133} z_1^2 z_2^{16} z_3^{170} = 0.$$

Hence the eleventh order neighborhood point $0_{32(11)}$, in the direction $x_1 = 0$, corresponds to the simple point $(z_1 = z_2 = 0)$.

Now apply the transformation M eleven times to the transformation

(40) $$\frac{X_1}{x_1^{17}} = \frac{X_2}{x_1^{14} x_2^2 x_3} = \frac{X_3}{x_1^{11} x_2^4 x_3^2} = \frac{X_4}{x_1^8 x_2^6 x_3^3} = \frac{X_5}{x_1^7 x_2 x_3^9}$$
$$= \frac{X_6}{x_1^5 x_2^8 x_3^4} = \frac{X_7}{x_1^4 x_2^3 x_3^{10}} = \frac{X_8}{x_1^7 x_2^{10} x_3^5} = \frac{X_9}{x_1 x_2^5 x_3^{11}} = \frac{X_{10}}{x_2^{17}}.$$

The result is

$$(41)\quad \frac{X_1}{z_1^{17}\, z_2^{171}} = \frac{X_2}{z_1^{14}\, z_2^{140}\, z_3^{34}} = \frac{X_3}{z_1^{11}\, z_2^{109}\, z_3^{68}} = \frac{X_4}{z_1^{8}\, z_2^{78}\, z_3^{102}}$$

$$= \frac{X_5}{z_1^{7}\, z_2^{62}\, z_3^{119}} = \frac{X_6}{z_1^{5}\, z_2^{47}\, z_3^{136}} = \frac{X_7}{z_1^{4}\, z_2^{31}\, z_3^{153}} = \frac{X_8}{z_1^{2}\, z_2^{16}\, z_3^{170}}$$

$$= \frac{X_9}{z_1\, z_3^{187}} = \frac{X_{10}}{z_2\, z_3^{187}}.$$

To allow for an all directional approach to the point $(z_1 = z_2 = 0)$, let $z_2 = k\, z_1$, and get

$$(42)\quad \frac{X_1}{k^{171}\, z_1^{187}} = \frac{X_2}{k^{140}\, z_1^{153}\, z_3^{34}} = \frac{X_3}{k^{109}\, z_1^{119}\, z_3^{68}} = \frac{X_4}{k^{78}\, z_1^{85}\, z_3^{102}}$$

$$= \frac{X_5}{k^{62}\, z_1^{68}\, z_3^{119}} = \frac{X_6}{k^{47}\, z_1^{51}\, z_3^{136}} = \frac{X_7}{k^{31}\, z_1^{34}\, z_3^{153}}$$

$$= \frac{X_8}{k^{16}\, z_1^{17}\, z_3^{170}} = \frac{X_9}{z_3^{187}} = \frac{X_{10}}{k\, z_3^{187}}.$$

As z_1 approaches the limit zero, we arrive at the equation of a tangent plane,

$$(43)\quad \begin{cases} X_{10} = k\, X_9 \\ X_1 = X_2 = X_3 = X_4 = X_5 = X_6 = X_7 = X_8 = 0. \end{cases}$$

Now consider the quadratic transformation

(L) $$x_1 : x_2 : x_3 = z_1z_3 : z_1z_2 : z_3^2$$

$(L)^{-1}$ $$z_1 : z_2 : z_3 = x_1^2 : x_2x_3 : x_1x_3.$$

Nine successive applications of L to equation (38) give

$$\begin{aligned}(44)\qquad & z_3^{153}\,(a_1\,z_1 + a_{59}\,z_2) + a_{170}\,z_1^{137}\,z_2^{17} + a_9\,z_1^{16}\,z_2^{2}\,z_3^{136} \\ & + a_{26}\,z_1^{31}\,z_2^{4}\,z_3^{119} + a_{99}\,z_1^{15}\,z_2^{3}\,z_3^{136} + a_{52}\,z_1^{46}\,z_2^{6}\,z_3^{102} \\ & + a_{87}\,z_1^{61}\,z_2^{8}\,z_3^{85} + a_{148}\,z_1^{30}\,z_2^{5}\,z_3^{119} + a_{133}\,z_1^{76}\,z_2^{10}\,z_3^{68} = 0.\end{aligned}$$

This indicates that the point in the ninth order neighborhood in the direction $x_2 = 0$, $0_{31(9)}$, corresponds to the simple point $(z_1 = z_2 = 0)$.

The transformation (40) under nine successive applications of L gives

$$\begin{aligned}(45)\qquad & \frac{X_1}{z_1\,z_3^{153}} = \frac{X_2'}{z_1^{16}\,z_2^{2}\,z_3^{136}} = \frac{X_3}{z_1^{31}\,z_2^{4}\,z_3^{119}} = \frac{X_4}{z_1^{46}\,z_2^{6}\,z_3^{102}} \\ & = \frac{X_5}{z_2\,z_3^{153}} = \frac{X_6}{z_1^{61}\,z_2^{8}\,z_3^{85}} = \frac{X_7}{z_1^{15}\,z_2^{3}\,z_3^{136}} = \frac{X_8}{z_1^{76}\,z_2^{10}\,z_3^{68}} \\ & = \frac{X_9}{z_1^{30}\,z_2^{5}\,z_3^{119}} = \frac{X_{10}}{z_1^{137}\,z_2^{17}}.\end{aligned}$$

Approach the point $(z_1 = z_2 = 0)$ from all directions by substituting $z_1 = k\, z_2$,

$$(46)\qquad \frac{X_1}{k\, z_3^{153}} = \frac{X_2}{k^{16}\, z_2^{17}\, z_3^{136}} = \frac{X_3}{k^{31}\, z_2^{34}\, z_3^{119}} = \frac{X_4}{k^{46}\, z_2^{51}\, z_3^{102}}$$

$$= \frac{X_5}{z_3^{153}} = \frac{X_6}{k^{61}\, z_2^{68}\, z_3^{85}} = \frac{X_7}{k^{15}\, z_2^{17}\, z_3^{136}} = \frac{X_8}{k^{76}\, z_2^{85}\, z_3^{68}}$$

$$= \frac{X_9}{k^{30}\, z_2^{34}\, z_3^{119}} = \frac{X_{10}}{k^{137}\, z_2^{153}} .$$

As z_2 tends to the limit zero, equations (46) give another tangent plane,

$$(47)\qquad \begin{cases} X_1 = k\, X_5 \\ X_2 = X_3 = X_4 = X_6 = X_7 = X_8 = X_9 = X_{10} = 0. \end{cases}$$

Note that when the transformation L is applied to equation (38) and then the transformation M is applied, the result is

$$(48)\qquad z_3^{34}\,(a_{59}\, z_1^2 + a_{99}\, z_1\, z_2 + a_{148}\, z_2^2) + a_1\, z_1^{11}\, z_2^8\, z_3^{17}$$

$$+ a_{170}\, z_1^{11}\, z_2^{25} + a_9\, z_1^{10}\, z_2^9\, z_3^{17} + a_{26}\, z_1^9\, z_2^{10}\, z_3^{17}$$

$$+ a_{52}\, z_1^8\, z_2^{11}\, z_3^{17} + a_{87}\, z_1^7\, z_2^{12}\, z_3^{17} + a_{133}\, z_1^6\, z_2^{13}\, z_3^{17} = 0.$$

This indicates that the second order neighborhood point, O_{312}, corresponds to the double point $(z_1 = z_2 = 0)$.

The application of L followed by M to the transformation (40) gives

$$(49) \quad \frac{X_1}{z_1^{11} z_2^{8} z_3^{17}} = \frac{X_2}{z_1^{10} z_2^{9} z_3^{17}} = \frac{X_3}{z_1^{9} z_2^{10} z_3^{17}} = \frac{X_4}{z_1^{8} z_2^{11} z_3^{17}}$$

$$= \frac{X_5}{z_1^{2} z_3^{34}} = \frac{X_6}{z_1^{7} z_2^{12} z_3^{17}} = \frac{X_7}{z_1 z_2 z_3^{34}} = \frac{X_8}{z_1^{6} z_2^{13} z_3^{17}}$$

$$= \frac{X_9}{z_2^{2} z_3^{34}} = \frac{X_{10}}{z_1^{11} z_2^{25}} .$$

Now approach the point $(z_1 = z_2 = 0)$ from all directions by making the substitution $z_2 = k\, z_1$,

$$(50) \quad \frac{X_1}{k^{8} z_1^{17} z_3^{17}} = \frac{X_2}{k^{9} z_1^{17} z_3^{17}} = \frac{X_3}{k^{10} z_1^{17} z_3^{17}} = \frac{X_4}{k^{11} z_1^{17} z_3^{17}}$$

$$= \frac{X_5}{z_3^{34}} = \frac{X_6}{k^{12} z_1^{17} z_3^{17}} = \frac{X_7}{k\, z_3^{34}} = \frac{X_8}{k^{13} z_1^{17} z_3^{17}} = \frac{X_9}{k^{2} z_3^{34}}$$

$$= \frac{X_{10}}{k^{25} z_1^{34}} .$$

As z_1 approaches the limit zero, a third tangent element is arrived at. It is

$$\text{(51)} \qquad \begin{cases} X_7^2 - X_5 X_9 = 0 \\ X_1 = X_2 = X_3 = X_4 = X_6 = X_8 = X_{10} = 0. \end{cases}$$

This surface is demonstrated to be a quadric cone, when investigated.

Thus, the following

Theorem 3: The tangent elements to ϕ at the point O'_{11} (0,0,0,0,0,0,0,0,0,0,1) are two planes (43), (47), and a quadric cone (51).

5. Multiplicities of Points O'_1, O'_{10}, and O'_{11} for Surface ϕ

The surface ϕ is of order 17. Two members of the family (18) intersect at O_1, $14 \cdot 1^2 + 7 \cdot 2^2 + 1 \cdot 3^2$ or 51 times [1, p. 30]. Thus, the system is of degree 289 - 51 or 238.† Since the curves (18) are related projectively to the hyperplanes of S_{10}, the multiplicity of the point O'_1 on ϕ is 51/17 or 3.

Two members of the family (28) intersect at O_2 in $1 \cdot 7^2 + 5 \cdot 2^2 + 2 \cdot 5^2$ or 119 fixed points. Since the system intersects in 289 - 119 variable points it is of degree 170. Also, the multiplicity of O'_{10} on ϕ is 119/17 or 7.

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†Degree is used in the same sense as Godeaux [5].

Two members of the family (38) intersect at O_3 in $1\cdot6^2 + 19\cdot1^2 + 1\cdot2^2 + 1\cdot3^2$ or 68 fixed points. The system is of degree 289 - 68 or 221, and the point O'_{11} on ϕ is of multiplicity 68/17 or 4.

6. Summary

The multiplicity of the curves (18), (28), and (38) at the points infinitely near O_1, O_2, and O_3 respectively have been investigated and quadratic transformations have been employed to examine the branch point images of these fundamental points. A pictorial diagram of these multiplicities is given in Figure 1.

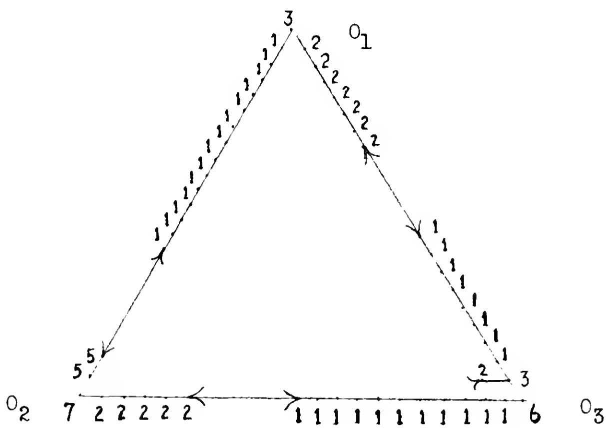

Figure 1

The tangent elements at O_1' constitute a plane and a quadric cone and the tangent elements at O_{10}' are a quintic cone and a quadric cone. The branch point O_{11}' is more interesting in that it has two tangent planes and a quadric cone.

CHAPTER III

PROJECTIONS OF THE SURFACE ϕ

1. Surface ϕ_1

The surface ϕ projects from the point O'_{10} (0,0,0,0,0,0,0,0,0,1,0) to the surface ϕ_1 in the space $X_{10} = 0$. The equations for the surface ϕ_1 are

$$(\phi_1) \qquad \begin{cases} \left\| \begin{matrix} X_8X_5 & X_6 & X_4 & X_9X_5 & X_3 & X_7 & X_2X_5 & X_2^2 \\ X_7X_9 & X_9 & X_7 & X_7X_{11} & X_5 & X_{11} & X_1X_{11} & X_1X_5 \end{matrix} \right\| = 0, \\ \\ X_{10} = 0. \end{cases}$$

Two members of the family (28) intersect in $1\cdot 7^2 + 5\cdot 2^2 + 2\cdot 5^2$ or 119 fixed points. Thus the order of ϕ_1 is $(289 - 119)/17$ or 10 (cf., Chapter II, Figure 1).

Now examine the family of curves (which pass through the point O_2 (0, 1, 0))

$$(52) \qquad a_{87}\, x_1^5\, x_2^8\, x_3^4 + a_{52}\, x_1^8\, x_2^6\, x_3^3 + a_{148}\, x_1\, x_2^5\, x_3^{11} + a_{26}\, x_1^{11}\, x_2^4\, x_3^2 + a_{99}\, x_1^4\, x_2^3\, x_3^{10} + a_9\, x_1^{14}\, x_2^2\, x_3 + a_{59}\, x_1^7\, x_2\, x_3^9 + a_1\, x_1^{17} + a_{171}\, x_3^{17} = 0.$$

Notice that the point O_2 is a nine tuple point.

Apply the quadratic transformation U twice to the family (52). The result is

$$(53)\qquad z_2^{34}\,(a_1 z_1^4 + a_9 z_1^3 z_3 + a_{26} z_1^2 z_3^2 + a_{52} z_1 z_3^3 + a_{87} z_3^4) + a_{148} z_1^{10} z_2^{17} z_3^{11} + a_{99} z_1^{11} z_2^{17} z_3^{10} + a_{59} z_1^{12} z_2^{17} z_3^{9} + a_{171} z_1^{21} z_3^{17} = 0.$$

Now apply the quadratic transformation V five successive times to (52) to obtain

$$(54)\qquad z_2^{85}\,(a_{148} z_1 + a_{171} z_3) + a_{87} z_1^5 z_2^{68} z_3^{13} + a_{52} z_1^8 z_2^{51} z_3^{27} + a_{26} z_1^{11} z_2^{34} z_3^{41} + a_{99} z_1^4 z_2^{68} z_3^{14} + a_9 z_1^{14} z_2^{17} z_3^{55} + a_{59} z_1^7 z_2^{51} z_3^{28} + a_1 z_1^{17} z_3^{69} = 0.$$

Also apply to (52) the quadratic transformation V, then U, and then V twice in succession. The result is

$$(55)\qquad z_2^{68}\,(a_{87} z_1 + a_{148} z_3) + a_{52} z_1^6 z_2^{51} z_3^{12} + a_{26} z_1^{11} z_2^{34} z_3^{24} + a_{99} z_1^5 z_2^{47} z_3^{13} + a_9 z_1^{16} z_2^{17} z_3^{36} + a_{59} z_1^{10} z_2^{34} z_3^{25} + a_1 z_1^{21} z_3^{48} + a_{171} z_1^4 z_2^{51} z_3^{14} = 0.$$

The three previous results indicate that the curves of system (52) have in common in the neighborhood of O_2:

(a) two successive four tuple points O_{21} and O_{211};

(b) a four tuple point O_{23};

(c) four successive simple points O_{233}, O_{2333}, O_{23333}, and O_{233333};

(d) three successive simple points O_{231}, O_{2313}, and O_{23133}.

Hence, two curves of system (52) intersect $9^2 + 3 \cdot 4^2 + 7 \cdot 1^2$ or 136 times at O_2. Therefore, the system (52) has degree 289 - 136 or 153. The sum of the multiplicity of O_{10}' for surface ϕ and the multiplicity of O_8' for ϕ_1 is 136/17 or 8. But O_{10}' is multiple of order 7 for ϕ. Hence O_8' is multiple of order 1 for ϕ_1.

Now in a manner similar to the material in Chapter II, apply the quadratic transformation U twice to the projectivity obtained from equation (52) and substitute $z_1 = k\, z_3$. As z_3 tends to the limit zero, the result is,

$$\text{(56)} \qquad \begin{cases} \left\| \begin{matrix} X_1 & X_2 & X_3 & X_4 \\ X_2 & X_3 & X_4 & X_6 \end{matrix} \right\| = 0, \\ \\ X_5 = X_7 = X_9 = X_{10} = X_{11} = 0. \end{cases}$$

Hence certain points of ϕ_1, infinitely near O_8', situated on (56), correspond to the points infinitely near O_{211}. Note that this is a projection of (33) to the space $X_{10} = 0$.

Apply the quadratic transformation V five successive times to the projectivity and substitute $z_3 = k z_1$. As z_1 approaches the limit zero, one gets the equations

$$(57) \qquad \begin{cases} X_{11} = k X_9 \\ X_1 = X_2 = X_3 = X_4 = X_5 = X_6 = X_7 = X_{10} = 0. \end{cases}$$

Hence certain points on ϕ_1, infinitely near O_8', situated on (57), correspond to the points infinitely near O_{233333}. Note that this is the projection of (37) to the space $X_{10} = 0$.

Now apply to the projectivity the quadratic transformation V, then U, and then V two successive times. Substitute $z_3 = k z_1$ and take the limit as z_1 approaches zero, and obtain

$$(58) \qquad \begin{cases} X_6 = k X_9 \\ X_1 = X_2 = X_3 = X_4 = X_5 = X_7 = X_{10} = X_{11} = 0. \end{cases}$$

Hence certain points of ϕ_1, infinitely near O_8', situated on (58), correspond to the points infinitely near O_{23133}.

Since (56) and (57) were projections of previous tangent elements, our new tangent element is the plane that projects (58) from O_8'. Hence, the following

<u>Theorem 4</u>: The surface ϕ_1 has a new tangent element

$$(59) \qquad \begin{cases} X_6 = k X_9 \\ X_1 = X_2 = X_3 = X_4 = X_5 = X_7 = X_{10} = X_{11} = 0 \end{cases}$$

at the point O_8'.

2. Surface ϕ_2

Project the surface ϕ_1 from the point O_8' (0,0,0,0,0,0,0,1,0,0,0) into the surface ϕ_2 in the space $X_8 = 0$, getting

$$(\phi_2) \qquad \begin{cases} \left\| \begin{matrix} X_6 & X_4 & X_9X_5 & X_3 & X_7 & X_2X_5 & X_2^2 \\ X_9 & X_7 & X_7X_{11} & X_5 & X_{11} & X_1X_{11} & X_1X_5 \end{matrix} \right\| = 0, \\ \\ X_{10} = X_8 = 0. \end{cases}$$

Two members of the family (52) intersect in $1\cdot 9^2 + 3\cdot 4^2 + 7\cdot 1^2$ or 136 fixed points. Thus the order of ϕ_2 is $(289 - 136)/17$ or 9.

Examine the family of curves which pass through O_2 (0, 1, 0),

$$(60) \qquad a_{52}\, x_1^8\, x_2^6\, x_3^3 + a_{148}\, x_1\, x_2^5\, x_3^{11} + a_{26}\, x_1^{11}\, x_2^4\, x_3^2 + a_{99}\, x_1^4\, x_2^3\, x_3^{10}$$
$$+ a_9\, x_1^{14}\, x_2^2\, x_3 + a_{59}\, x_1^7\, x_2\, x_3^9 + a_1\, x_1^{17} + a_{171}\, x_3^{17} = 0.$$

Note that O_2 is an eleven tuple point.

Apply the quadratic transformation U twice in succession to obtain

$$(61) \qquad z_2^{34}\,(a_1\, z_1^3 + a_9\, z_1^2\, z_3 + a_{26}\, z_1\, z_3^2 + a_{52}\, z_3^3) + a_{148}\, z_1^9\, z_2^{17}\, z_3^{11}$$
$$+ a_{99}\, z_1^{10}\, z_2^{17}\, z_3^{10} + a_{59}\, z_1^{11}\, z_2^{17}\, z_3^9 + a_{171}\, z_1^{20}\, z_3^{17} = 0.$$

Now apply the transformation V five successive times. The result is

$$(62) \qquad z_2^{85}\,(a_{148}\, z_1 + a_{171}\, z_3) + a_{52}\, z_1^8\, z_2^{51}\, z_3^{27} + a_{26}\, z_1^{11}\, z_2^{34}\, z_3^{41}$$
$$+ a_{99}\, z_1^4\, z_2^{68}\, z_3^{14} + a_9\, z_1^{14}\, z_2^{17}\, z_3^{55} + a_{59}\, z_1^7\, z_2^{51}\, z_3^{28}$$
$$+ a_1\, z_1^{17}\, z_2^{69} = 0.$$

One application of V and then six successive applications of U gives

$$(63)\qquad z_2^{51}(a_{52}z_1 + a_{148}z_3) + a_{26}z_1^{16}z_2^{34}z_3^2 + a_{99}z_1^{15}z_2^{34}z_3^3$$
$$+ a_9 z_1^{31}z_2^{17}z_3^4 + a_{59}z_1^{30}z_2^{17}z_3^5 + a_1 z_1^{46}z_3^6$$
$$+ a_{171}z_1^{29}z_2^{17}z_3^6 = 0.$$

All this indicates that the curves of (60) have in common in the neighborhood of O_2:

(a) two successive triple points O_{21} and O_{211};

(b) a double point O_{23};

(c) ten simple points O_{233}, O_{2333}, O_{23333}, O_{233333}, O_{231}, O_{2311}, O_{23111}, O_{231111}, $O_{2311111}$, and $O_{23111111}$.

Therefore, two curves of the system (60) intersect $1\cdot 11^2 + 2\cdot 3^2 + 1\cdot 2^2 + 10\cdot 1^2$ or 153 times at O_2. Thus, the system (60) has degree 289 - 153 or 136. The sum of the multiplicities of the points O_{10}' for ϕ, O_8' for ϕ_1, and O_6' for ϕ_2 must be 153/17 or 9. Hence O_6' is multiple of order one for ϕ_2.

Now apply the quadratic transformation U twice in succession to the projectivity obtained from (60). Then substitute $z_1 = k\, z_3$ and observe that the limit as z_3 goes to zero is

$$(64)\qquad \begin{cases} \left\| \begin{matrix} X_1 & X_2 & X_3 \\ X_2 & X_3 & X_4 \end{matrix} \right\| = 0, \\ \\ X_5 = X_7 = X_8 = X_9 = X_{10} = X_{11} = 0. \end{cases}$$

Hence to the points infinitely near O_{211} correspond certain points on Φ_2 of (64) near O_6'. Note that (64) is the projection of (56) to the space $X_8 = 0$.

Now apply V five successive times to the projectivity and substitute $z_3 = k\, z_1$. As z_1 tends to the limit zero the result is

$$(65) \qquad \begin{cases} X_{11} = k\, X_9 \\ X_1 = X_2 = X_3 = X_4 = X_5 = X_7 = X_8 = X_{10} = 0. \end{cases}$$

Note that (65) is the projection of (57) to the space $X_8 = 0$.

Apply V to the projectivity, and then apply U five successive times. Substitute $z_3 = k\, z_1$ and take the limit as z_1 tends to zero; the result is

$$(66) \qquad \begin{cases} X_9 = k\, X_4 \\ X_1 = X_2 = X_3 = X_5 = X_7 = X_8 = X_{10} = X_{11} = 0. \end{cases}$$

Surfaces (64) and (65) are projections of previous tangent elements. Thus, the additional tangent element to Φ_2 is the plane (66) as stated below.

<u>Theorem 5</u>: The surface Φ_2 has a new tangent element

$$(67) \qquad \begin{cases} X_9 = k\, X_4 \\ X_1 = X_2 = X_3 = X_5 = X_7 = X_8 = X_{10} = X_{11} = 0 \end{cases}$$

at the point O_6'.

3. Surface ϕ_3

Project the surface ϕ_2 from the point O'_6 (0,0,0,0,0,1,0,0,0,0,0) onto the space $X_6 = 0$ to obtain the surface, getting

$$(\phi_3) \qquad \begin{cases} \left\| \begin{matrix} X_4 & X_9X_5 & X_3 & X_7 & X_2X_5 & X_2^2 \\ X_7 & X_7X_{11} & X_5 & X_{11} & X_1X_{11} & X_1X_5 \end{matrix} \right\| = 0, \\ X_{10} = X_8 = X_6 = 0. \end{cases}$$

Two members of the family (60) intersect in $1\cdot 11^2 + 2\cdot 3^2 + 1\cdot 2^2 + 10\cdot 1^2$ or 153 fixed points. Thus, the order of ϕ_3 is (289 - 153)/17 or 8.

Consider the family of curves which pass through O_2 (0, 1, 0),

$$(68) \qquad a_{148}\, x_1\, x_2^5\, x_3^{11} + a_{26}\, x_1^{11}\, x_2^4\, x_3^2 + a_{99}\, x_1^4\, x_2^3\, x_3^{10} + a_9\, x_1^{14}\, x_2^2\, x_3 + a_{59}\, x_1^7\, x_2\, x_3^9 + a_1\, x_1^{17} + a_{171}\, x^{17} = 0.$$

Repeated application of the transformations U and V will show that members of the family (68) have in common at O_2:

(a) one triple point O_{21};

(b) one double point O_{211};

(c) thirteen simple points O_{213}, O_{2133}, ..., $O_{213(8)}$, O_{23}, O_{233}, O_{2333}, O_{23333}, and O_{233333}.

Hence, two members of the system (68) intersect in $1\cdot 12^2 + 1\cdot 3^2 + 1\cdot 2^2 + 13\cdot 1^2$ or 170 fixed points at O_2. Therefore, the system (68) has degree 289 - 170 or 119. This indicates that the sum of the multiplicities of O'_{10} for ϕ, O'_8 for ϕ_1, O'_6 for ϕ_2, and

O_4' for ϕ_3 is 170/17 or 10. Thus, O_4' is of multiplicity one for ϕ_3.

In a manner very much like those used before, apply the quadratic transformations U and V repeatedly to the projectivity and observe that:

(a) certain points near O_4' on ϕ_3 situated on

$$\begin{cases} X_2^2 - X_1X_3 = 0 \\ X_m = 0, \quad (m = 5, 6, 7, 8, 9, 10, 11) \end{cases} \tag{69}$$

correspond to the points infinitely near O_{211};

(b) certain other neighborhood points near O_4' on ϕ_3 situated on

$$\begin{cases} X_{11} = k\, X_9 \\ X_m = 0, \quad (m = 1, 2, 3, 5, 6, 7, 8, 10) \end{cases} \tag{70}$$

correspond to the points infinitely near O_{233333};

(c) and similarly, other points on ϕ_3 situated on

$$\begin{cases} X_3 = k\, X_9 \\ X_m = 0, \quad (m = 1, 2, 5, 6, 7, 8, 10, 11) \end{cases} \tag{71}$$

correspond to the points infinitely near $O_{213(8)}$.

Note that (69) is the projection of (64) to the space $X_6 = 0$ and that (70) is the projection of (65) to the space $X_6 = 0$.

The new tangent element is the plane projecting (71) from the point O_4'. Hence, the following

<u>Theorem 6</u>: The surface ϕ_3 has a new tangent element

$$(72)\qquad \begin{cases} X_3 = k\, X_9 \\ X_m = 0, \quad (m = 1, 2, 5, 6, 7, 8, 10, 11) \end{cases}$$

at the point O_4'.

4. Surface ϕ_4

Project the surface ϕ_3 from the point O_4' (0,0,0,1,0,0,0,0,0,0,0) to the space $X_4 = 0$ to obtain the surface

$$(\phi_4)\qquad \begin{cases} \left\| \begin{matrix} X_9X_5 & X_3 & X_7 & X_2X_5 & X_2^2 \\ X_7X_{11} & X_5 & X_{11} & X_1X_{11} & X_1X_5 \end{matrix} \right\| = 0, \\ X_{10} = X_8 = X_6 = X_4 = 0. \end{cases}$$

Two members of the family (68) intersect in $1 \cdot 12^2 + 1 \cdot 3^2 + 1 \cdot 2^2 + 13 \cdot 1^2$ or 170 fixed points at O_2 (0, 1, 0). Thus the order of ϕ_4 is (289 - 170)/17 or 7.

Examine the family of curves which pass through O_2,

$$(73)\qquad a_{26}\, x_1^{11}\, x_2^4\, x_3^2 + a_{99}\, x_1^4\, x_2^3\, x_3^{10} + a_9\, x_1^{14}\, x_2^2\, x_3 + a_{59}\, x_1^7\, x_2\, x_3^9 + a_1\, x_1^{17} + a_{171}\, x_3^{17} = 0.$$

Successive applications of the transformation U and V indicate that the members of the family (73) have in common at O_2:

(a) one four tuple point O_{23};

(b) three double points O_{21}, O_{211}, and O_{231};

(c) seven simple points O_{2313}, O_{23133}, O_{2311}, O_{23111}, O_{231111}, $O_{2311111}$, and $O_{231(6)}$.

Thus, two members of the system (73) meet at O_2 in $1\cdot 13^2 + 1\cdot 4^2 + 3\cdot 2^2 + 7\cdot 1^2$ or 204 fixed points, and the system has degree 289 - 204 or 85. This indicates that the sum of the orders of O'_{10} for ϕ, O'_8 for ϕ_1, O'_6 for ϕ_2, O'_4 for ϕ_3, and O'_9 for ϕ_4 is 204/17 or 12. Consequently, O'_9 for ϕ_4 is multiple of order two.

As before, apply the quadratic transformations to the projectivity, then make the necessary substitution of $z_1 = k\, z_3$ or $z_3 = k\, z_1$, and take the limit as z_3 or z_1 tends to zero respectively. This gives:

(a) certain points near O'_9 on ϕ_4 situated on

$$(74)\qquad \begin{cases} X_2^2 - X_1 X_3 = 0 \\ X_m = 0, \quad (m = 4, 5, 6, 7, 8, 10, 11) \end{cases}$$

correspond to the points infinitely near O_{211};

(b) similarly, certain points on ϕ_4 situated on

$$(75)\qquad \begin{cases} X_3 = k\, X_7 \\ X_m = 0, \quad (m = 1, 2, 4, 5, 6, 8, 10, 11) \end{cases}$$

correspond to the points infinitely near $O_{2311111}$,

(c) and other points on ϕ_4 situated on

(76)
$$\begin{cases} X_{11} = k\,X_7 \\ X_m = 0, \quad (m = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 8,\ 10) \end{cases}$$

correspond to the points infinitely near 0_{23133}.

The curve (74) is the projection of (69) to the space $X_4 = 0$. Hence, the following

<u>Theorem 7</u>: The surface ϕ_4 has the two new tangent elements

(77)
$$\begin{cases} X_3 = k\,X_7 \\ X_m = 0, \quad (m = 1,\ 2,\ 4,\ 5,\ 6,\ 8,\ 10,\ 11) \end{cases}$$

and

(78)
$$\begin{cases} X_{11} = k\,X_7 \\ X_m = 0, \quad (m = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 8,\ 10) \end{cases}$$

at the point O_9'.

5. Surface ϕ_5

Project the surface ϕ_4 from the point O_9' (0,0,0,0,0,0,0,0,1,0,0) to the space $X_9 = 0$ to obtain the surface

(ϕ_5)
$$\begin{cases} \left\| \begin{matrix} X_3 & X_7 & X_2X_5 & X_2^2 \\ X_5 & X_{11} & X_1X_{11} & X_1X_5 \end{matrix} \right\| = 0, \\ X_{10} = X_8 = X_6 = X_4 = X_9 = 0. \end{cases}$$

Two members of family (73) intersected in 204 fixed points at O_2. Thus the order of ϕ_5 is (289 - 204)/17 or 5.

Applications of the transformations U and V indicate that the family

$$(79)\qquad a_{99}\, x_1^4\, x_2^3\, x_3^{10} + a_9\, x_1^{14}\, x_2^2\, x_3 + a_{59}\, x_1^7\, x_2\, x_3^9 + a_1\, x_1^{17} + a_{171}\, x_3^{17} = 0,$$

has a group of multiple points at O_2. These multiple points are:

(a) one triple point O_{23};

(b) one double point O_{21};

(c) twelve simple points O_{231}, O_{2313}, O_{23133}, O_{211}, O_{213}, O_{2133}, ..., $O_{213(7)}$ and $O_{213(8)}$.

Hence, two members of the system (79) have in common at O_2, $1\cdot 14^2 + 1\cdot 3^2 + 1\cdot 2^2 + 12\cdot 1^2$ or 221 fixed points, and the degree of the system is 289 - 221 or 68. Delete from 221/17 the sum of the orders of O_{10}', O_8', O_6', O_4', and O_9'. Thus, the point O_3' is of multiplicity one for ϕ_5.

Applications of U and V to the projectivity give:

(a) the points on ϕ_5 situated on

$$(80)\qquad \begin{cases} X_1 = k\, X_2 \\ X_m = 0, \quad (m = 4, 5, 6, 7, 8, 9, 10, 11) \end{cases}$$

correspond to the points infinitely near O_{211};

(b) the points on ϕ_5 situated on

(81) $$\begin{cases} X_{11} = k\, X_7 \\ X_m = 0, \quad (m = 1, 2, 4, 5, 6, 8, 9, 10) \end{cases}$$

correspond to the points infinitely near O_{23133};

(c) the points on ϕ_5 situated on

(82) $$\begin{cases} X_2 = k\, X_7 \\ X_m = 0, \quad (m = 1, 4, 5, 6, 8, 9, 10, 11) \end{cases}$$

correspond to the points infinitely near $O_{213(8)}$.

Note that the surface (80) is the projection of (74) to the space $X_9 = 0$, and that (81) is the projection of (76) to the same space.

Hence, the following

Theorem 8: The surface ϕ_5 has a tangent new element

(83) $$\begin{cases} X_2 = k\, X_7 \\ X_m = 0, \quad (m = 1, 4, 5, 6, 8, 9, 10, 11) \end{cases}$$

at the point O_3'.

6. Surface ϕ_6

Project the surface ϕ_5 from the point O_3' (0,0,1,0,0,0,0,0,0,0,0) to the space $X_3 = 0$ to obtain the surface

(ϕ_6) $$\begin{cases} \left\| \begin{matrix} X_7 & X_2X_5 & X_2^2 \\ X_{11} & X_1X_{11} & X_1X_5 \end{matrix} \right\| = 0, \\ X_{10} = X_8 = X_6 = X_4 = X_9 = X_3 = 0. \end{cases}$$

Two members of the family (79) intersected in 221 variable points at O_2. Thus, the order of ϕ_6 is (289 - 221)/17 or 4.

Now use the transformations U and V to find the multiple points at O_2 of the family

$$(84) \qquad a_9\, x_1^{14}\, x_2^2\, x_3 + a_{59}\, x_1^7\, x_2\, x_3^9 + a_1\, x_1^{17} + a_{171}\, x_3^{17} = 0.$$

The multiple points are:

(a) seven double points O_{23}, O_{231}, ... , $O_{23(5)}$, and $O_{23(6)}$;

(b) two simple points O_{21} and O_{211}.

Consequently, two members of the system (84) have in common at O_2, $1\cdot15^2 + 7\cdot2^2 + 2\cdot1^2$ or 255 fixed points, and the degree of the system is 289 - 255 or 34. Since the sum of the orders of O'_{10}, O'_8, O'_6, O'_4, O'_9, and O'_3 is 13, the value 255/17, or 15, implies that the multiplicity of O'_7 for the surface ϕ_6 is 2.

Applications of U and V to the projectivity give:

(a) the points on ϕ_6 situated on

$$(85) \qquad \begin{cases} X_1 = k\, X_2 \\ X_m = 0, \quad (m = 3, 4, 5, 6, 8, 9, 10, 11) \end{cases}$$

correspond to the points infinitely near O_{211};

(b) the points on ϕ_6 situated on

$$(86) \qquad \begin{cases} X_5^2 - X_2X_{11} = 0 \\ X_m = 0, \quad (m = 1, 3, 4, 6, 8, 9, 10) \end{cases}$$

correspond to the points infinitely near $O_{23(6)}$.

The surface (85) is the projection of the surface (80) to the space $X_3 = 0$. Hence, the following

Theorem 9: The surface ϕ_6 has a new tangent element

(87)
$$\begin{cases} X_5^2 - X_2X_{11} = 0 \\ X_m = 0, \quad (m = 1, 3, 4, 6, 8, 9, 10) \end{cases}$$

at the point O_7'.

7. Surface ϕ_7

The surface ϕ_6 projects from the point O_7' (0,0,0,0,0,0,1,0,0,0,0) to the space $X_7 = 0$ to a new surface

(ϕ_7)
$$\begin{cases} X_5^2 - X_2X_{11} = 0 \\ X_{10} = X_8 = X_6 = X_4 = X_9 = X_3 = X_7 = 0. \end{cases}$$

Two members of the family (84) intersected in 255 fixed points at O_2. Thus, the order of ϕ_7 is (289 - 255)/17 or 2.

The transformations U and V establish the multiple points at O_2 for the family

(88) $$a_{59}\, x_1^7\, x_2\, x_3^9 + a_1\, x_1^{17} + a_{171}\, x_3^{17} = 0.$$

These are sixteen simple points O_{21}, O_{213}, O_{2133}, $\cdots$, $O_{213(8)}$, O_{23}, O_{231}, O_{2311}, $\cdots$, $O_{231(5)}$, and $O_{231(6)}$. Thus, two members of the system (88) have in common at O_2, $1 \cdot 16^2 + 16 \cdot 1^2$ or 272 fixed points, and the degree of the system is 289 - 272 or 17. The multiplicities of O_{10}',

O'_8, O'_6, O'_4, O'_9, O'_3, O'_7, and O'_2 total to 272/17 or 16. Therefore O'_2 is a simple point for ϕ_7.

Apply U and V to the projectivity and observe that:

(a) the points on ϕ_7 situated on

(89) $$\begin{cases} X_5 = k\,X_{11} \\ X_m = 0, \quad (m = 1, 3, 4, 6, 7, 8, 9, 10) \end{cases}$$

correspond to the points infinitely near $O_{23111111}$;

(b) the points on ϕ_7 situated on

(90) $$\begin{cases} X_1 = k\,X_5 \\ X_m = 0, \quad (m = 3, 4, 6, 7, 8, 9, 10, 11) \end{cases}$$

correspond to the points infinitely near $O_{213(8)}$.

The surface (89) is the projection of (86) to the space $X_7 = 0$. Hence, the following

<u>Theorem 10</u>: The surface ϕ_7 has a new tangent element

$$\begin{cases} X_1 = k\,X_5 \\ X_m = 0, \quad (m = 3, 4, 6, 7, 8, 9, 10, 11) \end{cases}$$

at the point O'_2.

8. Summary

A sequence of projected surfaces was described and the tangent elements investigated. The orders of these surfaces were arrived at and the multiplicities of the points calculated.

The following figures are an outline of the multiple points that the generating curves have at the O_2 vertex of the triangle of reference in the plane.

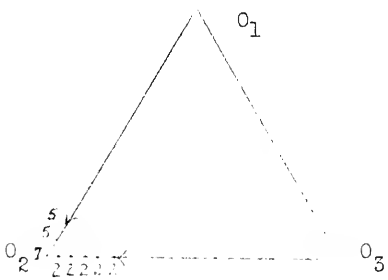

Figure 2

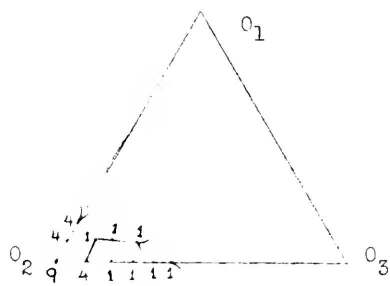

Figure 3

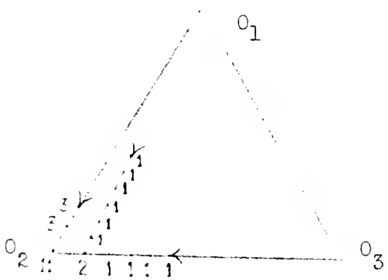

Figure 4

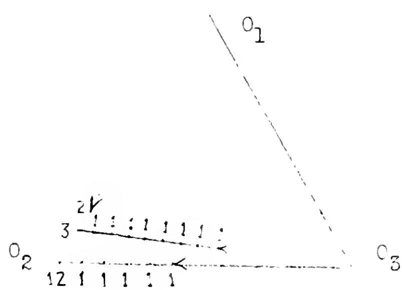

Figure 5

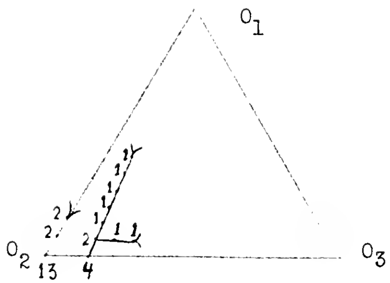

Figure 6

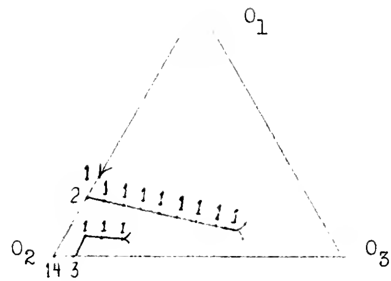

Figure 7

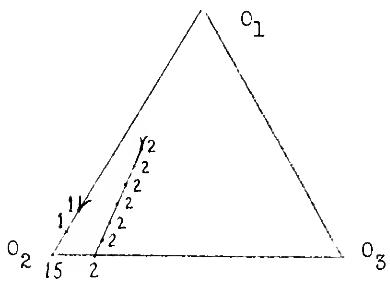

Figure 8

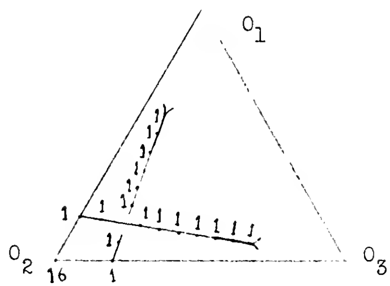

Figure 9

The following chart lists the various results of this chapter and some information on ϕ from Chapter II.

Surface	Order of Surface	Point on Surface	Multiplicity of Point	Tangent Element to Surface at Point
ϕ	17	O'_{10}	7	$\left\Vert\begin{matrix} X_1 & X_2 & X_3 & X_4 & X_6 \\ X_2 & X_3 & X_4 & X_6 & X_8 \end{matrix}\right\Vert = 0$ and $X_m = 0,\ (m = 5,7,9,11)$ $X_9^2 - X_8X_{11} = 0$ and $X_m = 0,\ (m = 1,2,3,4,5,6,7)$
ϕ_1	10	O'_8	1	$X_6 = k\ X_9$ and $X_m = 0,\ (m = 1,2,3,4,5,7,10,11)$
ϕ_2	9	O'_6	1	$X_9 = k\ X_4$ and $X_m = 0,\ (m = 1,2,3,5,7,8,10,11)$
ϕ_3	8	O'_4	1	$X_3 = k\ X_9$ and $X_m = 0,\ (m = 1,2,5,6,7,8,10,11)$

Surface	Order of Surface	Point on Surface	Multiplicity of Point	Tangent Element to Surface at Point
Φ_4	7	O_9'	2	$X_3 = k\ X_9$ and $X_m = 0,\ (m = 1,2,4,5,6,8,10,11)$ $X_{11} = k\ X_7$ and $X_m = 0,\ (m = 1,2,3,4,5,6,8,10)$
Φ_5	5	O_3'	1	$X_2 = k\ X_7$ and $X_m = 0,\ (m = 1,4,5,6,8,9,10,11)$
Φ_6	4	O_7'	2	$X_5^2 - X_2X_{11} = 0$ and $X_m = 0,\ (m = 1,3,4,6,8,9,10)$
Φ_7	2	O_2'	1	$X_1 = k\ X_5$ and $X_m = 0,\ (m = 3,4,6,7,8,9,10,11)$

The author realizes that the results obtained in this chapter only begin to identify the information that is obtainable about the projected surfaces. Further study will undoubtedly yield many other fascinating truths, illuminating the facts of this chapter.

CHAPTER IV

A RATIONAL SURFACE F IN S_{11}

To a certain plane curve shown below, of order seventeen, and which is not invariant under H corresponds on Φ a curve of order two hundred eighty-nine. This curve is cut out on Φ by a seventeenth order hypersurface. Furthermore, the coefficients of the equations of the latter surface are functions of the coefficients of the equation of the plane curve considered.

In order to see this, consider the plane curve of order seventeen,

(92) $$\theta_1 \equiv \Sigma\, c_{ijk}\, x_1^i\, x_2^j\, x_3^k = 0,$$

where

$$i + j + k = 17.$$

Apply H sixteen times in succession to (92); this gives

$$\theta_n = \Sigma\, E^{w(n)}\, c_{ijk}\, x_1^i\, x_2^j\, x_3^k = 0,$$

where

$$i + j + k = 17$$

$$n = 2, 3, \ldots, 17,$$

and $w(n)$ is the remainder when $(n - 1)(j + 15k)$ is divided by 17.

The curve,

(93) $$\theta_1\, \theta_2\, \theta_3 \ldots \theta_{17} = 0,$$

corresponds to a curve C on Φ, where C is in birational correspondence with each of the curves $\theta_m = 0$ ($m = 1, 2, \ldots, 17$). That is, to a point

of C corresponds seventeen points of the plane with one of the seventeen points on each of the seventeen curves considered.

The curve (93) meets a curve of (1) in two hundred eighty-nine groups of I_{17}. This implies that the hyperplane related to (1) intersects C in 289 points. Hence, C is of order 289.

Let us vary θ_1 in a continuous manner in its plane until its equation becomes equal to (1). The corresponding C varies on ϕ and reduces to the section of ϕ by the hyperplane,

$$(94) \qquad a_1 X_1 + a_9 X_2 + a_{26} X_3 + a_{52} X_4 + a_{59} X_5 + a_{87} X_6 + a_{99} X_7$$

$$+ a_{133} X_8 + a_{148} X_9 + a_{170} X_{10} + a_{171} X_{11} = 0,$$

counted seventeen times. That is, the section of ϕ is made by the reducible hypersurface of order 17,

$$(95) \qquad (a_1 X_1 + a_9 X_2 + a_{26} X_3 + \dots + a_{171} X_{11})^{17} = 0.$$

This implies that the curves C are cut out on ϕ by seventeenth order hypersurfaces.

Now, consider $\theta_1 = 0$ varying in the plane and becoming equation (2). The curve (93) becomes

$$(96) \qquad (g(x_1, x_2, x_3))^{17} = 0,$$

and the curve C becomes a curve A counted seventeen times. Consequently, A must be cut out on ϕ by a hypersurface of order seventeen.

By simplifying (96) and applying T one arrives at the following equation for the hypersurface

$$(97) \qquad \Psi(X_1, X_2, \ldots , X_{11}) \equiv (g(x_1, x_2, x_3))^{17} \equiv a_{17}^{17} X_1^{12} X_{11}^{5}$$

$$+ a_{17}^{16} a_{29} X_1^{11} X_2 X_9 X_{11}^{4} + \ldots + a_{169}^{17} X_{10}^{8} X_{11}^{9} = 0.$$

The fact that the x_i's ($i = 1, 2, 3$) group together into factors of X_i $(i = 1, \ldots , 11)$ can be demonstrated by solving certain equations relating the exponents of x_i $(i = 1, 2, 3)$ obtained from possible powers of terms of $g(x_1, x_2, x_3)$ to the exponents of x_i $(i = 1, 2, 3)$ obtained from possible factors of X_i $(i = 1, \ldots , 11)$.

Take a surface F in S_{11} whose equations are:

$$X_{12}^{17} = \Psi(X_1, X_2, \ldots , X_{11})$$

$$(F) \qquad \left\| \begin{matrix} X_{10}X_1 & X_8X_5 & X_6 & X_4 & X_9X_5 & X_3 & X_7 & X_2X_5 & X_2^2 \\ X_4X_6 & X_7X_9 & X_9 & X_7 & X_7X_{11} & X_5 & X_{11} & X_1X_{11} & X_1X_5 \end{matrix} \right\| = 0.$$

Now the author demonstrates that F is a rational surface. To do this, a projective correspondence is set up between the plane and F using the following transformation T'.

$$(T') \quad \frac{X_1}{x_1^{17}} = \frac{X_2}{x_1^{14}\,x_2^2\,x_3} = \frac{X_3}{x_1^{11}\,x_2^4\,x_3^2} = \frac{X_4}{x_1^8\,x_2^6\,x_3^3} = \frac{X_5}{x_1^7\,x_2\,x_3^9}$$

$$= \frac{X_6}{x_1^5\,x_2^8\,x_3^4} = \frac{X_7}{x_1^4\,x_2^3\,x_3^{10}} = \frac{X_8}{x_1^2\,x_2^{10}\,x_3^5} = \frac{X_9}{x_1\,x_2^5\,x_3^{11}} = \frac{X_{10}}{x_2^{17}}$$

$$= \frac{X_{11}}{x_3^{17}} = \frac{X_{12}}{g(x_1,\, x_2,\, x_3)} = \frac{1}{\rho}\,.$$

This transformation orders to each point of the plane a unique point of F. It needs to be shown that the converse is true. A development of T'^{-1} will show this and thus show that F is a rational surface.

The first of the following ten equations comes directly from T'. The others are derived with successive multiplications by $X_4\,X_5\,X_6$ and applications of T'.

$$(98) \quad g(x_1,\, x_2,\, x_3) = \rho X_{12}$$

$$\begin{aligned}(99) \quad & a_{124}\,X_4\,X_5\,X_6\,x_1^2\,x_2\,x_3^{14} + a_{169}\,X_4\,X_5\,X_6\,x_2^8\,x_3^9 \\ & + a_{42}\,X_3\,X_4\,X_5\,x_1^3\,x_2^6\,x_3^8 + a_{75}\,X_4\,X_6\,X_9\,x_1^{12}\,x_3^5 \\ & + a_{96}\,X_4\,X_6\,X_9\,x_1^{10}\,x_2^7 + a_{143}\,X_4\,X_6\,X_8\,x_1^6\,x_2^4\,x_3^7 \\ & + a_{119}\,X_3\,X_5\,X_9\,x_1^4\,x_2^{11}\,x_3^2 + a_{17}\,X_2\,X_4\,X_9\,x_1^9\,x_2^2\,x_3^6 \\ & + a_{29}\,X_2\,X_4\,X_9\,x_1^7\,x_2^9\,x_3 + a_{58}\,X_3\,X_4\,X_5\,x_1\,x_2^{13}\,x_3^3 \\ & = \rho\,X_4\,X_5\,X_6\,X_{12}.\end{aligned}$$

(100) $$a_{124} X_4^2 X_5 X_6^2 X_{11} x_1^9 x_2^2 x_3^6 + a_{169} X_4^2 X_5 X_6^2 X_{11} x_1^7 x_2^9 x_3$$
$$+ a_{75} X_1 X_4^2 X_6^2 X_9 x_1^2 x_2 x_3^{14} + a_{96} X_1 X_4^2 X_6^2 X_9 x_2^8 x_3^9$$
$$+ a_{143} X_4^2 X_6^2 X_8 X_9 x_1^{12} x_3^5 + a_{17} X_2^2 X_4^2 X_5 X_{11} x_1 x_2^{13} x_3^3$$
$$+ a_{29} X_3^2 X_4^2 X_9^2 x_1^{10} x_2^7 + a_{58} X_3 X_4^2 X_5^2 X_{10} x_1^6 x_2^4 x_3^7$$
$$+ a_{42} X_2 X_4^2 X_5^2 X_9 x_1^4 x_2^{11} x_3^2 + a_{119} X_4^2 X_5^2 X_6^2 x_1^3 x_2^6 x_3^8$$
$$= \rho X_4^2 X_5^2 X_6^2 X_{12}.$$

(101) $$a_{124} X_2 X_4^3 X_5 X_6^3 X_{11} x_1^2 x_2 x_3^{14} + a_{169} X_2 X_4^3 X_5 X_6^3 X_{11} x_2^8 x_3^9$$
$$+ a_{75} X_1 X_4^3 X_6^3 X_9 X_{11} x_1^9 x_2^2 x_3^6 + a_{96} X_1 X_4^3 X_5 X_6^3 X_{11} x_1 x_2^{13} x_3^3$$
$$+ a_{143} X_4^3 X_5^2 X_6^3 X_9 x_1^7 x_2^9 x_3 + a_{17} X_2^2 X_4^3 X_5^2 X_{10} X_{11} x_1^6 x_2^4 x_3^7$$
$$+ a_{29} X_3^3 X_4^3 X_5 X_9^2 x_1^4 x_2^{11} x_3^2 + a_{58} X_3 X_4^3 X_5^3 X_9 X_{10} x_1^{10} x_2^7$$
$$+ a_{42} X_2 X_4^3 X_5^3 X_6 X_8 x_1^3 x_2^6 x_3^8 + a_{119} X_4^2 X_5^3 X_6^2 X_8^2 x_1^{12} x_3^5$$
$$= \rho X_4^3 X_5^3 X_6^3 X_{12}.$$

$$(102)\quad a_{124} X_2 X_4^4 X_5 X_6^4 X_{11}^2 x_1^9 x_2^2 x_3^6 + a_{169} X_2 X_4^4 X_5 X_6^4 X_{11}^2 x_1^7 x_2^9 x_3$$

$$+ a_{75} X_1 X_2 X_4^4 X_6^4 X_9 X_{11} x_1^2 x_2 x_3^{14}$$

$$+ a_{96} X_1 X_4^4 X_5 X_6^4 X_8 X_{11} x_1^6 x_2^4 x_3^7$$

$$+ a_{143} X_4^4 X_5^2 X_6^4 X_8 X_9 x_1^{12} x_3^5 + a_{17} X_2^2 X_4^4 X_5^4 X_{10} X_{11} x_1^4 x_2^{11} x_3^2$$

$$+ a_{29} X_3^3 X_4^4 X_5^3 X_9 X_{10} x_1^3 x_2^6 x_3^8 + a_{58} X_2 X_3 X_4^4 X_5^4 X_9 X_{10} x_1 x_2^{13} x_3^3$$

$$+ a_{42} X_2 X_3 X_4^4 X_5^4 X_8^2 x_2^8 x_3^9 + a_{119} X_3 X_4^3 X_5^4 X_6 X_8^2 X_9 x_1^{10} x_2^7$$

$$= \rho X_4^4 X_5^4 X_6^4 X_{12}.$$

$$(103)\quad a_{124} X_2^2 X_4^5 X_5 X_6^5 X_{11}^2 x_1^2 x_2 x_3^{14} + a_{169} X_2^2 X_4^5 X_5 X_6^5 X_{11}^2 x_2^8 x_3^9$$

$$+ a_{75} X_1 X_2 X_4^5 X_6^5 X_9 X_{11}^2 x_1^9 x_2^2 x_3^6$$

$$+ a_{96} X_1 X_4^5 X_5 X_6^5 X_8 X_9 X_{11} x_1^{12} x_3^5$$

$$+ a_{143} X_1 X_4^5 X_5^2 X_6^5 X_9 X_{11} x_1^4 x_2^{11} x_3^2$$

$$+ a_{17} X_2^2 X_4^5 X_5^5 X_8 X_{10} X_{11} x_1^7 x_2^9 x_3$$

$$+ a_{29} X_3^3 X_4^5 X_5^5 X_9 X_{10} x_1 x_2^{13} x_3^3 + a_{58} X_2 X_3 X_4^5 X_5^5 X_9 X_{10}^2 x_1^6 x_2^4 x_3^7$$

$$+ a_{42} X_2 X_3 X_4^5 X_5^5 X_8^3 x_1^3 x_2^6 x_3^8 + a_{119} X_3 X_4^4 X_5^5 X_6^2 X_8^2 X_9 x_1^{10} x_2^7$$

$$= \rho X_4^5 X_5^5 X_6^5 X_{12}.$$

$$(104)\quad a_{124} X_2^2 X_4^6 X_5 X_6^6 X_{11}^3 x_1^9 x_2^2 x_3^6 + a_{169} X_2^2 X_4^6 X_5 X_6^6 X_{11}^3 x_1^7 x_2^9 x_3$$
$$+ a_{75} X_1 X_2^2 X_4^6 X_6^6 X_9 X_{11}^2 x_1^2 x_2 x_3^{14}$$
$$+ a_{96} X_1^2 X_4^6 X_5 X_6^6 X_9 X_{11}^2 x_1^4 x_2^{11} x_3^2$$
$$+ a_{143} X_1 X_4^6 X_5^2 X_6^6 X_9^2 X_{11} x_1^{10} x_2^7 + a_{17} X_2^2 X_4^6 X_5^6 X_8 X_{10}^2 X_{11} x_1^{12} x_3^5$$
$$+ a_{29} X_3^3 X_4^6 X_5^6 X_9 X_{10}^2 x_1^6 x_2^4 x_3^7 + a_{58} X_2 X_3^2 X_4^6 X_5^6 X_9 X_{10}^2 x_2^8 x_3^9$$
$$+ a_{42} X_2^2 X_4^6 X_5^6 X_8^4 x_1^3 x_2^6 x_3^8 + a_{119} X_2 X_3 X_4^5 X_5^6 X_6^2 X_8^2 X_9 x_1 x_2^{13} x_3^3$$
$$= \rho X_4^6 X_5^6 X_6^6 X_{12}.$$

$$(105)\quad a_{124} X_2^3 X_4^7 X_5 X_6^7 X_{11}^3 x_1^2 x_2 x_3^{14} + a_{169} X_2^3 X_4^7 X_5 X_6^7 X_{11}^3 x_2^8 x_3^9$$
$$+ a_{75} X_1 X_2^2 X_4^7 X_6^7 X_9 X_{11}^3 x_1^9 x_2^2 x_3^6 + a_{96} X_1^2 X_4^7 X_5 X_6^7 X_9^2 X_{11}^2 x_1^{10} x_2^7$$
$$+ a_{143} X_1 X_3 X_4^7 X_5^2 X_6^7 X_9^2 X_{11} x_1^6 x_2^4 x_3^7$$
$$+ a_{17} X_2^3 X_4^7 X_5^7 X_8 X_{10}^2 X_{11} x_1^3 x_2^6 x_3^8 + a_{29} X_3^3 X_4^7 X_5^7 X_{10}^3 X_{11} x_1^{12} x_3^5$$
$$+ a_{58} X_2 X_3^2 X_4^7 X_5^7 X_9^2 X_{10}^2 x_1^4 x_2^{11} x_3^2 + a_{42} X_2^2 X_4^7 X_5^7 X_8^4 X_9 x_1^7 x_2^9 x_3$$
$$+ a_{119} X_1 X_3 X_4^6 X_5^7 X_6^2 X_8^3 X_9 x_1 x_2 x_3^{13}$$
$$= \rho X_4^7 X_5^7 X_6^7 X_{12}.$$

(106) $$a_{124} X_2^3 X_4^8 X_5 X_6^8 X_{11}^4 x_1^9 x_2^2 x_3^6 + a_{169} X_2^3 X_4^8 X_5 X_6^8 X_{11}^4 x_1^7 x_2^9 x_3$$
$$+ a_{75} X_1 X_2^3 X_4^8 X_6^8 X_9 X_{11}^3 x_1^2 x_2 x_3^{14}$$
$$+ a_{96} X_1^3 X_4^8 X_5 X_6^8 X_9^2 X_{11}^2 x_2^8 x_3^9$$
$$+ a_{143} X_1 X_3 X_4^8 X_5^2 X_6^8 X_9^3 X_{11} x_1^{12} x_3^5$$
$$+ a_{17} X_2^3 X_4^8 X_5^8 X_8^2 X_{10}^2 X_{11} x_1^6 x_2^4 x_3^7$$
$$+ a_{29} X_2 X_3^3 X_4^8 X_5^8 X_{10}^3 X_{11} x_1^3 x_2^6 x_3^8$$
$$+ a_{58} X_3^3 X_4^8 X_5^8 X_8 X_9^2 X_{10}^2 x_1^{10} x_2^7 + a_{42} X_2^2 X_3 X_4^8 X_5^8 X_8^4 X_9 x_1 x_2^{13} x_3^3$$
$$+ a_{119} X_1 X_3 X_4^7 X_5^8 X_6^2 X_8^4 X_9 x_1^4 x_2^{11} x_3^2 = \rho X_4^8 X_5^8 X_6^8 X_{12}.$$

(107) $$a_{124} X_2^4 X_4^9 X_5 X_6^9 X_{11}^4 x_1^2 x_2 x_3^{14} + a_{169} X_2^4 X_4^9 X_5 X_6^9 X_{11}^4 x_2^8 x_3^9$$
$$+ a_{75} X_1 X_2^3 X_4^9 X_6^9 X_9 X_{11}^4 x_1^9 x_2^2 x_3^6$$
$$+ a_{96} X_1^3 X_4^9 X_5 X_6^9 X_9^2 X_{11}^3 x_1^7 x_2^9 x_3 + a_{143} X_1^2 X_4^9 X_5^2 X_6^9 X_9^4 X_{11} x_1^{12} x_3^5$$
$$+ a_{17} X_2^3 X_4^9 X_5^9 X_8^2 X_9 X_{10}^2 X_{11} x_1^{10} x_2^7$$
$$+ a_{29} X_2 X_3^3 X_4^9 X_5^9 X_8 X_{10}^3 X_{11} x_1^6 x_2^4 x_3^7$$
$$+ a_{58} X_3^4 X_4^9 X_5^9 X_8 X_9^2 X_{10}^2 x_1^4 x_2^{11} x_3^2$$
$$+ a_{42} X_2^3 X_4^9 X_5^9 X_8^4 X_9 X_{10} x_1^3 x_2^6 x_3^8$$
$$+ a_{119} X_1 X_3 X_4^9 X_5^9 X_6^2 X_8^4 X_9 x_1 x_2^{13} x_3^3 = \rho X_4^9 X_5^9 X_6^9 X_{12}.$$

Note that the previous ten equations are linear in $x_1^{12}\, x_3^5$, $x_1^{10}\, x_2^7$, ... , and $x_2^8\, x_3^9$, i.e., the terms of $g(x_1, x_2, x_3)$. Thus, Cramer's rule can be easily used to solve for these expressions.

The ten equations give

$$x_1^6\, x_2^4\, x_3^7 = \frac{\rho\, \Delta_5\, X_{12}}{\Delta}, \tag{108}$$

where Δ and Δ_i $(i = 1, 2, \ldots, 10)$ are the determinants that are functions of the ten constant coefficients and of X_i $(i = 1,2,3,4,5,6,8,9,10,11)$. Now from T' one obtains

$$x_3^{17} = \rho\, X_{11}. \tag{109}$$

This gives with (108) the following

$$\frac{x_1^6\, x_2^4}{x_3^{10}} = \frac{\Delta_5\, X_{12}}{\Delta\, X_{11}}. \tag{110}$$

In like manner the ten equations give

$$x_1^{12}\, x_3^5 = \frac{\rho\, \Delta_1\, X_{12}}{\Delta}, \tag{111}$$

and T' yields

$$x_1^{17} = \rho\, X_1. \tag{112}$$

Now combine the above two equations,

$$\frac{x_3^5}{x_1^5} = \frac{\Delta_1\, X_{12}}{\Delta\, X_1}. \tag{113}$$

Next, combine (110) and (113) to get

$$\frac{x_2^4}{x_1^4} = \frac{\Delta_1^2 \, \Delta_5 \, X_{12}^3}{\Delta^3 \, X_1^2 \, X_{11}} . \tag{114}$$

The transformation T' yields

$$x_1^7 \, x_2 \, x_3^9 = \rho X_5 . \tag{115}$$

Also, the ten equations give

$$x_2^8 \, x_3^9 = \frac{\rho \, \Delta_{10} \, X_{12}}{\Delta} . \tag{116}$$

These two equations combine to give

$$\frac{x_1^7}{x_2^7} = \frac{\Delta \, X_5}{\Delta_{10} \, X_{12}} . \tag{117}$$

Now combine (114) and (117) to give

$$\frac{x_2}{x_1} = \frac{\Delta_1^4 \, \Delta_5^2 \, X_5 \, X_{12}^5}{\Delta^5 \, \Delta_{10} \, X_1^4 \, X_{11}^2} . \tag{118}$$

The transformation T' also yields

$$x_1 \, x_2^5 \, x_3^{11} = \rho \, X_9 , \tag{119}$$

and the ten equations give

$$x_1 \, x_2^{13} \, x_3^3 = \frac{\rho \, \Delta_9 \, X_{12}}{\Delta} . \tag{120}$$

These two equations combine to give

(121)
$$\frac{x_3^8}{x_2^8} = \frac{\Delta\ X_9}{\Delta_9\ X_{12}}\ .$$

From the transformation T' obtain

(122)
$$x_1^2\ x_2^{10}\ x_3^5 = \rho\, X_8.$$

The ten equations give

(123)
$$x_1^2\ x_2\ x_3^{14} = \frac{\rho\, \Delta_8\ X_{12}}{\Delta}\ .$$

The above two equations combine to give

(124)
$$\frac{x_2^9}{x_3^9} = \frac{\Delta\ X_8}{\Delta_8\ X_{12}}\ .$$

Now (121) and (124) combine to give

(125)
$$\frac{x_2}{x_3} = \frac{\Delta^2\ X_8\ X_9}{\Delta_8\ \Delta_9\ X_{12}^2}\ .$$

Finally combine (118) and (125) to give the inverse of the transformation T'. It is

(T'^{-1})
$$\frac{x_1}{\Delta^7\ \Delta_{10}\ X_1^4\ X_8\ X_9\ X_{11}^2} = \frac{x_2}{\Delta^2\ \Delta_1^4\ \Delta_5^2\ X_5\ X_8\ X_9\ X_{12}^5}$$

$$= \frac{x_3}{\Delta_1^4\ \Delta_5^2\ \Delta_8\ \Delta_9\ X_5\ X_{12}^7}\ .$$

Hence, there is a one-to-one correspondence between the plane and the surface F, even though there is a one-to-seventeen correspondence between the surface ϕ and the plane and also between ϕ and F.

Conclusion

Using an homography, an involution of period seventeen was generated; and certain surfaces obtained from this involution were investigated. A family of plane curves invariant under this involution was projected, by means of the transformation T, to the hyperplanes of a space of ten dimensions (S_{10}). From this a surface ϕ, with points on it in a one-to-seventeen correspondence to the points of the plane, was arrived at. Then a study of the tangent elements at three branch points was carried out.

The next section of the study constituted a series of projections of this surface ϕ. The surfaces arrived at by successive projections were ϕ_i ($i = 1, 2, \ldots, 7$). Then certain tangent elements at selected points on the various surfaces were exhibited.

By adding to the transformation T used previously, an additional coordinate X_{12} proportional to the function $g(x_1, x_2, x_3)$, a projectivity T', mapping the points of the plane onto the points of a surface F in S_{11}, is established. Now each point of the plane is mapped onto a point on the surface F. By exhibiting the inverse of T' each point of F is mapped onto a point on the plane. Hence, the surface F is rational.

APPENDIX I

A METHOD OF FINDING THE ORDER OF A QUINTIC TANGENT CONE

There are various techniques of determining the order of a surface. The particular method illustrated here employs the definition from Woods [26, p. 390].

Examine as an example the surface (33). The equations of this surface combined with those of two general hyperplanes will give an homogeneous equation in two homogeneous variables. The degree of this final equation will be the order of the original surface.

Solve simultaneously the equations:

(126) $X_5 = X_7 = X_9 = X_{11} = 0$

(127) $X_2^2 - X_1X_3 = 0$

(128) $X_3^2 - X_2X_4 = 0$

(129) $X_4^2 - X_3X_6 = 0$

(130) $X_6^2 - X_4X_8 = 0$

(131) $\Sigma\, A_i\, X_i = 0 \; (i = 1, 2, \ldots , 11)$

(132) $\Sigma\, B_j\, X_j = 0 \; (j = 1, 2, \ldots , 11).$

The combination of equations (126), (127), (131), and (132) will give after simplification

$$(133)\quad (A_1B_{10} - A_{10}B_1)\, X_2^2 + (A_2B_{10} - A_{10}B_2)\, X_2X_3 + (A_3B_{10} - A_{10}B_3)\, X_3^2$$
$$+ (A_4B_{10} - A_{10}B_4)\, X_3X_4 + (A_6B_{10} - A_{10}B_6)\, X_3X_6$$
$$+ (A_8B_{10} - A_{10}B_8)\, X_3X_8 = 0.$$

Now substitute equation (128) to eliminate X_2 and then use (129) to remove X_3,

$$(134)\quad (A_1B_{10} - A_{10}B_1)\, X_4^4 + (A_2B_{10} - A_{10}B_2)\, X_4^3X_6 + (A_3B_{10} - A_{10}B_3)\, X_4^2X_6^2$$
$$+ (A_4B_{10} - A_{10}B_4)\, X_4X_6^3 + (A_6B_{10} - A_{10}B_6)\, X_6^4$$
$$+ (A_8B_{10} - A_{10}B_8)\, X_6^3X_8 = 0.$$

Now employ (130) to arrive at

$$(135)\quad (A_1B_{10} - A_{10}B_1)\, X_6^5 + (A_2B_{10} - A_{10}B_2)\, X_6^4X_8 + (A_3B_{10} - A_{10}B_3)\, X_6^3X_8^2$$
$$+ (A_4B_{10} - A_{10}B_4)\, X_6^2X_8^3 + (A_6B_{10} - A_{10}B_6)\, X_6X_8^4$$
$$+ (A_8B_{10} - A_{10}B_8)\, X_8^5 = 0.$$

Note that the solution of the fifth degree equation (135) indicates that the tangent element is a quintic cone.

APPENDIX II

A METHOD OF INVESTIGATING A FOURTEENTH ORDER NEIGHBORHOOD

In the investigation of involutions that involve large values of p such that $E^p = 1$, there arises the problem of applying a quadratic transformation repetitively to a large equation. For example, in the study of O_1', a quadratic transformation R had to be applied fourteen times to a seventeenth degree equation, cf., Chapter II.

The problem is not quite as difficult as the reader might first expect. The use of homogeneous coordinates makes the computation slightly less involved.

The following is a description of how a pattern develops. Observe that a term $z_1^i z_2^j z_3^k$ under R goes into $z_1^{2i+j} z_2^{j+k} z_3^k$. The a_{170} term stops any factoring of z_3's in the simplification. Thus for any given term the k, or the exponent of z_3, remains constant under applications of R. The a_9 term allows only one z_2 to be factored out in the simplification. Hence, for a given term the z_2 exponent will increase by the constant value $k - 1$ under each application. Now the a_{171} term allows only $z_1^{i+j+k-17}$ to be factored out. This final result after simplification is $z_1^{2i+j-i-j-k+17} z_2^{j+k-1} z_3^k$ or $z_1^{i-k+17} z_2^{j+k-1} z_3^k$. For a given term z_1 increases by the constant $17 - k$ and z_2 increases by the constant $k - 1$ for each application. This pattern develops only after one complete application of R.

The above explanation is not meant to be a proof that a similar constant increase pattern develops under certain types of quadratic transformations applied to any homogeneous equation, even though a related theorem might conceivably be constructed. The explanation is included here simply because it happened in all the applications of this paper and it was a considerable time saver.

The following chart of numbers is included as a display of the exponents of the z's under fourteen applications of the transformation R to equation (18).

a_9 x_1^{14}	x_2^2	x_3	a_{26} x_1^{11}	x_2^4	x_3^2	a_{52} x_1^8	x_2^6	x_3^3	a_{59} x_1^7	x_2	x_3^9	a_{87} x_1^5	x_2^8	x_3^4
14	2	1	11	4	2	8	6	3	7	1	9	5	8	4
30	0	1	26	3	2	22	6	3	15	7	9	18	9	4
46	0	1	41	4	2	36	8	3	23	15	9	31	12	5
62	0	1	56	5	2	50	10	3	31	23	9	44	15	4
78	0	1	71	6	2	64	12	3	39	31	9	57	18	4
94	0	1	86	7	2	78	14	3	47	39	9	70	21	4
110	0	1	101	8	2	92	16	3	55	47	9	83	24	4
126	0	1	116	9	2	106	18	3	63	55	9	96	27	4
142	0	1	131	10	2	120	20	3	71	63	9	109	30	4
158	0	1	146	11	2	134	22	3	79	71	9	122	33	4
174	0	1	161	12	2	148	24	3	87	79	9	135	36	4
190	0	1	176	13	2	162	26	3	95	87	9	148	39	4
206	0	1	191	14	2	176	28	3	103	95	9	161	42	4
222	0	1	206	15	2	190	30	3	111	103	9	174	45	4
238	0	1	221	16	2	204	32	3	119	111	9	187	48	4

$a_{99}\ x_1^4\ x_2^3\ x_3^{10}$			$a_{133}\ x_1^2\ x_2^{10}\ x_3^5$			$a_{148}\ x_1\ x_2^5\ x_3^{11}$			$a_{170}\ x_2^{17}$			$a_{171}\ x_3^{17}$		
4	3	10	2	10	5	1	5	11	0	17	0	0	0	17
11	10	10	14	12	5	7	13	11	17	14	0	0	14	17
18	19	10	26	16	5	13	23	11	34	13	0	0	30	17
25	28	10	38	20	5	19	33	11	51	12	0	0	46	17
32	37	10	50	24	5	25	43	11	68	11	0	0	62	17
39	46	10	62	28	5	31	53	11	85	10	0	0	78	17
46	55	10	74	32	5	37	63	11	102	9	0	0	94	17
53	64	10	86	36	5	43	73	11	119	8	0	0	110	17
60	73	10	98	40	5	49	83	11	136	7	0	0	126	17
67	82	10	110	44	5	55	93	11	153	6	0	0	142	17
74	91	10	122	48	5	61	103	11	170	5	0	0	158	17
81	100	10	134	52	5	67	113	11	187	4	0	0	174	17
88	109	10	146	56	5	73	123	11	204	3	0	0	190	17
95	118	10	158	60	5	79	133	11	221	2	0	0	206	17
102	127	10	170	64	5	85	143	11	238	1	0	0	222	17

APPENDIX III

A METHOD OF DEMONSTRATING THE EXISTENCE OF $\Psi(X_1, X_2, \ldots, X_{11})$

The following is one way of justifying that the terms of $(g(x_1, x_2, x_3))^{17}$ can be expressed in terms of the X_i's $(i = 1, \ldots, 11)$ of the transformation T.

Suppose that the $a_{17}, a_{29}, \ldots, a_{169}$ terms are raised to the powers $b_1, b_2, \ldots, b_{10}$ respectively and one factors out $\prod X_i^j$ where $i = 1, 2, \ldots, 11$ and $j = d_1, d_2, \ldots, d_{11}$. Then $(g(x_1, x_2, x_3))^{17}$ will be expressible as products of X_i's if all possible combinations of integral values of $b_i = 0, 1, \ldots, 17$, and $\Sigma\, b_j = 17$ are such that the equations (136), (137), and (138) have integral solutions of d_i's where $\Sigma\, d_i = 17$. Equations (136), (137), and (138) are the conditions that cause the exponents of the x_1, x_2, and x_3 to be the same in the powers and the factors.

(136) $$12\, b_1 + 10\, b_2 + 9\, b_3 + 7\, b_4 + 6\, b_5 + 4\, b_6 + 3\, b_7 + 2\, b_8 + b_9$$
$$= 17\, d_1 + 14\, d_2 + 11\, d_3 + 8\, d_4 + 7\, d_5 + 5\, d_6 + 4\, d_7 + 2\, d_8 + d_9.$$

(137) $$7\, b_2 + 2\, b_3 + 9\, b_4 + 4\, b_5 + 11\, b_6 + 6\, b_7 + b_8 + 13\, b_9 + 8\, b_{10}$$
$$= 2\, d_2 + 4\, d_3 + 6\, d_4 + d_5 + 8\, d_6 + 3\, d_7 + 10\, d_8 + 5\, d_9 + 17\, d_{10}.$$

$$(138) \quad 5\,b_1 + 6\,b_3 + b_4 + 7\,b_5 + 2\,b_6 + 8\,b_7 + 14\,b_8 + 3\,b_9 + 9\,b_{10}$$

$$= d_2 + 2\,d_3 + 3\,d_4 + 9\,d_5 + 4\,d_6 + 10\,d_7 + 5\,d_8 + 11\,d_9 + 17\,d_{11}.$$

Now use $\Sigma\, b_i = 17$ and $\Sigma\, d_i = 17$ to eliminate b_8 and d_{11} in equations (136), (137), and (138). The results are

$$(139) \quad 34 + 10\,b_1 + 8\,b_2 + 7\,b_3 + 5\,b_4 + 4\,b_5 + 2\,b_6 + b_7 - b_9 - 2\,b_{10}$$

$$= 17\,d_1 + 14\,d_2 + 11\,d_3 + 8\,d_4 + 7\,d_5 + 5\,d_6 + 4\,d_7 + 2\,d_8 + d_9,$$

$$(140) \quad 17 - b_1 + 6\,b_2 + b_3 + 8\,b_4 + 3\,b_5 + 10\,b_6 + 5\,b_7 + 12\,b_9 + 7\,b_{10}$$

$$= 2\,d_2 + 4\,d_3 + 6\,d_4 + d_5 + 8\,d_6 + 3\,d_7 + 10\,d_8 + 5\,d_9 + 17\,d_{10},$$

$$(141) \quad 51 + 9\,b_1 + 14\,b_2 + 8\,b_3 + 13\,b_4 + 7\,b_5 + 12\,b_6 + 6\,b_7 + 11\,b_9$$

$$+ 5\,b_{10} = 17\,d_1 + 16\,d_2 + 15\,d_3 + 14\,d_4 + 8\,d_5 + 13\,d_6 + 7\,d_7$$

$$+ 12\,d_8 + 6\,d_9 + 17\,d_{10}.$$

Observe that the above three equations are not independent, i.e., equations (139) and (140) added together give equation (141). Consequently, if (139) and (140) are satisfied then (141) will automatically be satisfied.

To exhibit that (139) and (140) have a common solution assume that the b_2, b_3, b_4, b_5, b_6, b_7 are each equated respectively to d_4, d_5, d_6, d_7, d_8, d_9. The problem now is reduced to showing that

equations (142) and (143) have a common solution.

$$(142) \qquad 34 + 10\ b_1 - b_9 - 2\ b_{10} = 17\ d_1 + 14\ d_2 + 11\ d_3,$$

$$(143) \qquad 17 - b_1 + 12\ b_9 + 7\ b_{10} = 2\ d_2 + 4\ d_3 + 17\ d_{10}.$$

Now eliminate b_9 in the above two equations to obtain

$$(144) \qquad 25 + 7\ b_1 - b_{10} = 12\ d_1 + 10\ d_2 + 8\ d_3 + d_{10}.$$

The author has examined individually all the possible variations of b_1 and b_{10} in equation (144). They are considerably too many to be listed here.

The above analysis is included to explain to the reader how the number of test situations are substantially reduced.

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BIOGRAPHICAL SKETCH

John Willis Kenelly, Jr., was born at Bogalusa, Louisiana, on November 22, 1935. He attended the public schools of the City of Bogalusa and was graduated from Bogalusa High School in May, 1953. Immediately thereafter he entered Southeastern Louisiana College, Hammond, Louisiana, and completed the requirements for the degree of Bachelor of Science with a major in mathematics in August, 1956. This degree was conferred with honors at the following commencement exercises on May 25, 1957. He entered the University of Mississippi in the fall of 1956 and received the degree of Master of Science with a major in mathematics on August 18, 1957. Subsequently he entered the University of Florida in the fall of 1957 and has pursued graduate studies since that time, with the exception of the summer of 1960.

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